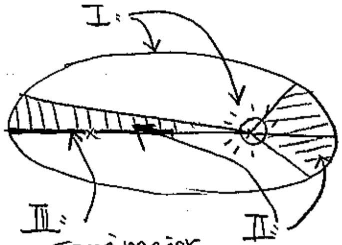
Wolfson Ch 8 Gravity

1. Kepler 三大行星運動定律 (Fig 8.1)



$T^2 \propto (a/2)^3$

Based solely on observation.
i.e. Kepler knew <u>how</u> the planets moved, but not <u>why</u>.
This "why" is the universal gravitation by Newton:
兩個質量為 m_1, m_2 的質點間的吸引力大小為

$$F = G\frac{m_1 m_2}{r^2}, \quad r = \text{separation of } m_1 \text{ and } m_2.$$

G: 萬有引力常數 $= 6.67 \times 10^{-11}\ N \cdot m^2 \cdot kg^{-2}$.

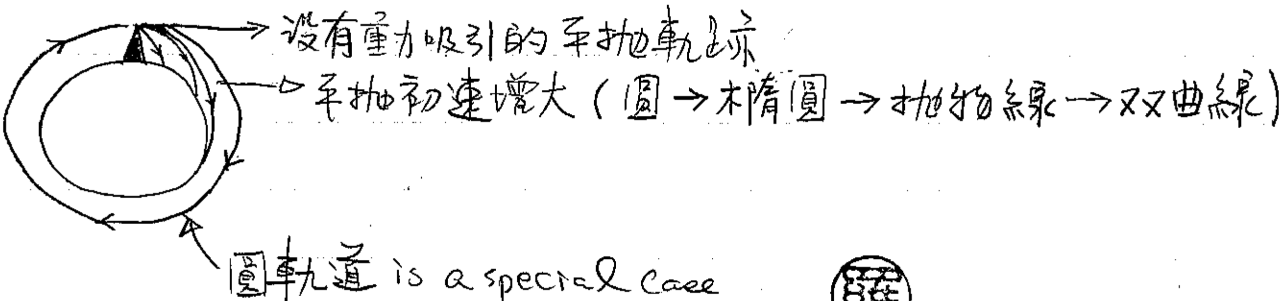
質點性質的適用性:
when $r \gg$ size of m_1 and m_2 或 m_1, m_2 具有球狀對稱分布質量。如太空間 ($r \gg$ size of Earth) vs. 地球 (↑)

2. 軌道運動

定量分析圓形軌道, 定性描述一般軌道。

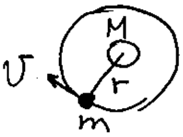
• 圓形軌道
Newton's genius: hold 住 moon 進行圓軌道的力也拉 apple 往下掉的力相同.

Fig 8.5 牛頓的想像 exp. (no air-friction): 水平初速 v_0



Why satellite's path is not a straight line? ⇒ 重力作用!

In a UCM, 向心力 $= ma_r = \frac{mv^2}{r} = G\frac{mM}{r^2}$



(assume $M >> m$, so M is at rest)

$\therefore v = \sqrt{\frac{GM}{r}}$ (軌道 speed)

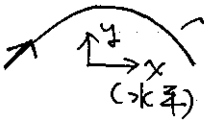
and $T = \frac{2\pi r}{v}$, $\therefore \left(\frac{2\pi r}{T}\right)^2 = \frac{GM}{r} \Rightarrow T^2 = \frac{4\pi^2}{GM} r^3$, 此為

Kepler's III law for special case — 圓軌道。

Note: v and T 和 m 無關, i.e. all 物体具有相同的重力加速度。
If 衛星的 $T = 24$ h ⇒ 同步衛星 (geosychronous satellite)

- 橢圓軌道 (elliptical orbit)

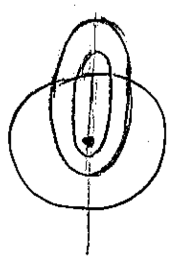
Ch.3 在忽略地表曲率及 $g(r)=$ constant 時, 拋体軌跡為拋物線 (parabola)



Actually:

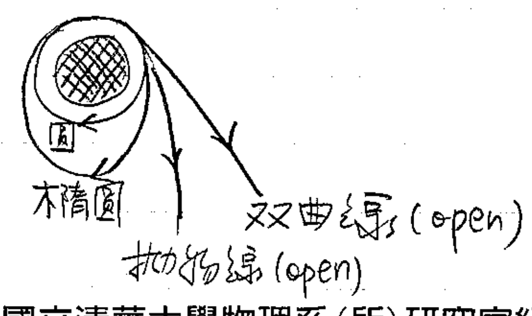
→ 在忽略 air 阻力時, 拋体軌跡為橢圓, 地球中心為其一個焦點, 在拋体軌跡 $\ll R_E$, 無法区分橢圓和拋物線

Fig. 8.7



圓軌道 and 橢圓軌道為封閉軌道。

⇒ open orbits vs. closed orbits Fig 8.8



隨發射衛星的初速↑, 衛星的軌道: 圓 → 橢圓 → 拋物線 → 双曲線 (hyperbola)

3、Gravitational energy

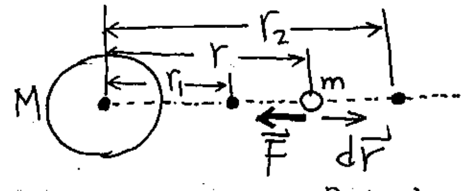
發射同步衛星到同步軌道,需 energy mgh ? ⇒ No. ∵ g is not constant. (mgh 只有在地表附近才成立).

衛星的 total energy $E = K + U$, $U = ?$

∘ $U_G(r) = ?$

When m moves from $r_1 \to r_2$,

$\Delta U_G = ?$



(r, r_1, r_2 自M中心量起)

$$\Delta U_G = -\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = -\int_{r_1}^{r_2} [-F(r)\,dr] \quad \text{where } F(r) \doteq G\frac{Mm}{r^2}.$$

$$\therefore \Delta U_G = GMm\int_{r_1}^{r_2}\frac{dr}{r^2} = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = U_G(r_2) - U_G(r_1)$$

$$\therefore U_G(r) = -\frac{GMm}{r} + \text{常數}.$$

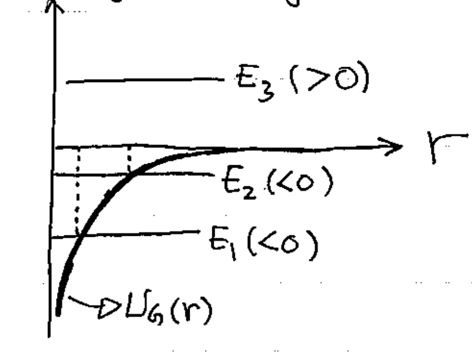
(↳ Assess: when $r_1 < r_2$, $\Delta U_G > 0$, 與地表的 $\Delta U_g = mg(r_2 - r_1)$ 正負号相符.)

∵ 常數 $= U_G(r\to\infty)$, ∴ Set 常數 $= 0 = U_G(r\to\infty)$ 為 U_G 參考點.

$$\therefore U_G(r) = -\frac{GMm}{r} \quad (\text{重力位能})$$

(cf: $U_g(h)$ = 地表附近的重力位能, with $h \ll R_E$)

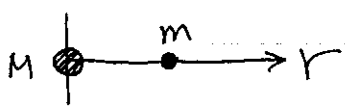
energy (Fig 8.11)



$E = K + U$

when $E < 0$, m has a turning point
⇒ closed orbit.

when $E > 0$, m has no turning point
⇒ open orbit.



- Escape speed

From Fig 8.11, $E = K + U < 0$ 時, m is bound to M.
在 $E \geq 0$ 才能脫離M的束縛, 此時所需的動能 $K = ?$
所需的 min. K occurs at $E = 0 = K + U = \frac{1}{2}mv^2 - \frac{GMm}{r}$
此時的 speed named as "escape speed" v_{esc}

$$\therefore v_{esc} = \sqrt{\frac{2GM}{r}}$$ 與m無關.

地表的 $v_{esc} = 11.2$ km/s, 在更高軌道則更小。

- 圓軌道的 energy

$$a_r = \frac{v^2}{r} = \frac{GM}{r^2} \quad \therefore v^2 = \frac{GM}{r}$$

$$\text{and } K = \frac{1}{2}mv^2 = \frac{GMm}{2r} = -\frac{1}{2}U$$

$$\therefore \boxed{E = K + U = -K = \frac{1}{2}U = -\frac{GMm}{2r}}$$

→ 軌道低, $K\uparrow$, $v\uparrow$; 高軌道則 $v\downarrow$, $T\uparrow$.

在橢圓軌道, $E = K + U = $ constant, 但 U 和 K 互相消長. 近焦點處 $K\uparrow$, $U\downarrow$; 反之 $K\downarrow$, $U\uparrow$。

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4. 重力場

超距力 (action-at-a-distance force) 曾困擾物理及哲學家.

⇒ 場的觀念

M 建立一個 field, m 再與 field 交互作用、∴ 此場只與 M 有關, 與 m 無關.

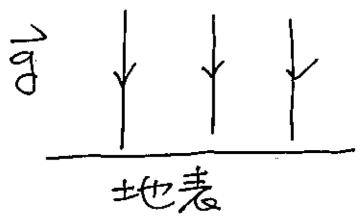
$$\rightarrow F(r) = G\frac{Mm}{r^2} \Rightarrow \text{場 of } M = GM/r^2$$

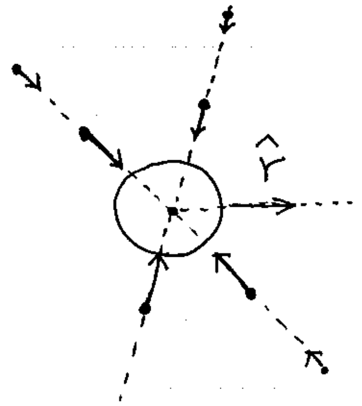
∴ 重力場 = 單位質量所受的萬有引力

$$\vec{F}_G(r) = -\frac{GMm}{r^2}\hat{r} \quad (\text{作用在 } m \text{ 上})$$

∴ M所形成的 field $\vec{g}(r) = \frac{\vec{F}_G(r)}{m} = -\frac{GM}{r^2}\hat{r}$

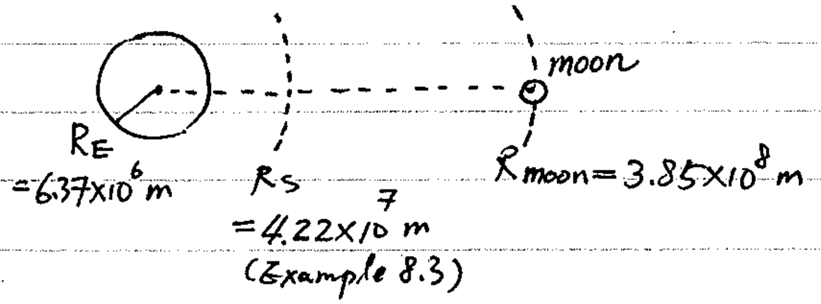
地表附近 $(R_E + h \sim R_E)$ $\vec{g}(R_E) = -\frac{GM}{R_E^2}\hat{r} = -g\hat{r} = \text{constant}.$





Wolfson Example 8.4

m 的物质 from 地表→同步軌道；再 from 同步軌道→Moon，
比較此2过程中，克服地球重力位能所需作的功各是多少？



State 1：地球表面
State 2：同步軌道
State 3：到達Moon軌道

State 1 → State 2：需作功 $W_{12} = \boxed{U_2 - U_1}$ （U only）

State 2 → State 3：需作功 $W_{23} = U_3 - U_2$ （U only!）

∴在只考慮重力位能時，則

$$W_{12} = U_2 - U_1 = GMm\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = GMm\left(\frac{1}{R_E} - \frac{1}{R_S}\right)$$

$$= 5.8\times10^{11}\ \text{J}$$

$$W_{23} = U_3 - U_2 = GMm\left(\frac{1}{r_2} - \frac{1}{r_3}\right) = GMm\left(\frac{1}{R_S} - \frac{1}{R_{moon}}\right)$$

$$= 9.2\times10^{9}\ \text{J}$$

If 如果問：最少需要花多少energy？（從地表発射）

$\Rightarrow E = K + U$，$\Delta E = E_f - E_i$

State 1：$K_1 \doteqdot 0$，$U_1 = -\frac{GMm}{R_E}$，$E_1 = K_1 + U_1 = -U_1/2 = -\frac{GMm}{2R_E}$

State 2：$\underset{\neq 0}{K_2}$，$U_2 = -\frac{GMm}{R_S}$，$E_2 = -\frac{1}{2}U_2 = -\frac{GMm}{2R_S}$

（$K_1 \doteqdot 0$ for 自転）

$$\therefore \Delta E_{12} = E_2 - E_1 = \frac{GMm}{2}\left(\frac{1}{R_E} - \frac{1}{R_S}\right)$$

Similarly $\Delta E_{23} = E_3 - E_2 = \frac{GMm}{2}\left(\frac{1}{R_S} - \frac{1}{R_{moon}}\right)$

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