(E3.5) A system is represented by a block diagram as shown in Figure E3.5. Write the state equations in the form of Equations (3.16) and (3.17).

$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}.$ $\mathbf{y} = \mathbf{Cx} + \mathbf{Du},$

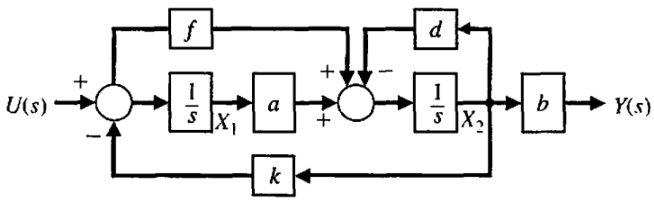


FIGURE E3.5 Block diagram.

Ans.

$\frac{1}{s}(-fkX_2(s) - dX_2(s) + aX_1(s) + fU(s)) = X_2(s) \Rightarrow \dot{x}_2 = -(fk+d)x_2 + ax_1 + fu$

$\frac{1}{s}\left(U(s) - kX_2(s)\right) = X_1(s) \Rightarrow \dot{x}_1 = -kx_2 + u$

- Therefore, $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \mathbf{y} = \mathbf{Cx} + \mathbf{Du}$

$$\mathbf{A} = \begin{bmatrix} 0 & -k \\ a & -(fk+d) \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 \\ f \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 0 & b \end{bmatrix}, \mathbf{D} = [0], \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \mathbf{u} = [u], \mathbf{y} = [y]$$

(E3.6) A system is represented by Equation (3.16), where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(a) Find the matrix $\mathbf{\Phi}(t)$

(b) For the initial conditions x1(0) = x2(0) = l, find **x**(t).

Ans.

(a) The state transition matrix is $\mathbf{\Phi}(t) = e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \frac{1}{2!}\mathbf{A}^2t^2 + \cdots$

$$\mathbf{A}^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0, \text{thus } \mathbf{A}^2 = \mathbf{A}^3 = \mathbf{A}^4 = \cdots = 0$$

$$\Rightarrow \mathbf{\Phi}(t) = e^{\mathbf{A}t} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

(b) The state at any time t ≥ 0 is given by $\mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0)$

Since x1(0) = x2(0) = 1

$$\Rightarrow \mathbf{x}(t) = \begin{bmatrix} 1+t \\ 1 \end{bmatrix}, x_1(t) = 1 + t, x_2(t) = 1$$

(E3.9) A multi-loop block diagram is shown in Figure E3.9. The state variables are denoted by x1 and x2.

(a) Determine a state variable representation of the closed-loop system where the output is denoted by y(t) and the input is r(t)

(b) Determine the characteristic equation.

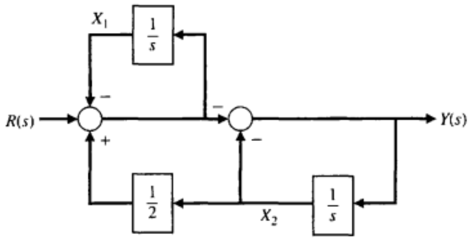


FIGURE E3.9 Multi-loop feedback control system.

Ans.

(a) $X_1(s) = \frac{1}{s}\left(-X_1(s) + R(s) + \frac{1}{2}X_2(s)\right) \Rightarrow \dot{x}_1 = -x_1 + \frac{1}{2}x_2 + r$

$X_2(s) = \frac{1}{s}\left(-\left(-X_1(s) + R(s) + \frac{1}{2}X_2(s)\right) - X_2(s)\right) \Rightarrow \dot{x}_2 = x_1 - \frac{3}{2}x_2 - r$

$Y(s) = -X_2(s) - \left(-X_1(s) + R(s) + \frac{1}{2}X_2(s)\right) \Rightarrow y = x_1 - \frac{3}{2}x_2 - r$

- In state-variable form

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0.5 \\ 1 & -1.5 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}\mathbf{r}, \mathbf{y} = \begin{bmatrix} 1 & -1.5 \end{bmatrix}\mathbf{x} + [-1]\mathbf{r}$$

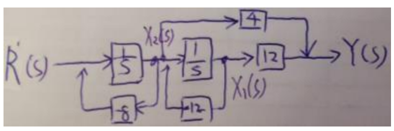
(b) The characteristic equation

$$\det\begin{bmatrix} -1-s & 0.5 \\ 1 & -1.5-s \end{bmatrix} = 0 \rightarrow s^2 + \frac{5}{2}s + 1 = (s+\frac{1}{2})(s+2) = 0$$

(E3.11) Determine a state variable representation for the system described by the transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{4(s+3)}{(s+2)(s+6)}$$

Ans.



$$\frac{Y(s)}{R(s)} = \frac{4(s+3)}{(s+2)(s+6)} = \frac{4s+12}{s^2+8s+12} = \frac{4s^{-1}+12s^{-2}}{1+8s^{-1}+12s^{-2}}$$

$X_1(s) = \frac{1}{s}\left(-12X_1(s) + X_2(s)\right) \Rightarrow \dot{x}_1 = -12x_1 + x_2$

$X_2(s) = \frac{1}{s}\left(-8X_2(s) + R(s)\right) \Rightarrow \dot{x}_2 = -8x_2 + r$

$Y(s) = 12X_1(s) + 4X_2(s) \Rightarrow y = 12x_1 + 4x_2$

- $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r}, \mathbf{y} = \mathbf{C}\mathbf{x}$

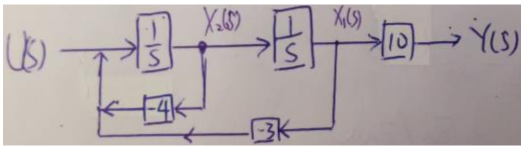
  $\Rightarrow \mathbf{A} = \begin{bmatrix} -12 & 1 \\ 0 & -8 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 12 & 4 \end{bmatrix}$

(E3.19) A single-input, single-output system has the matrix equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\mathbf{u}, \mathbf{y} = \begin{bmatrix} 10 & 0 \end{bmatrix}\mathbf{x}$$

Determine the transfer function G(s) = Y(s)/U(s).

Ans.



$$G(s) = \frac{Y(s)}{U(s)} = \frac{10s^{-2}}{1+4s^{-1}+3s^{-2}} = \frac{10}{s^2+4s+3}$$

(P3.12) A system is described by its transfer function

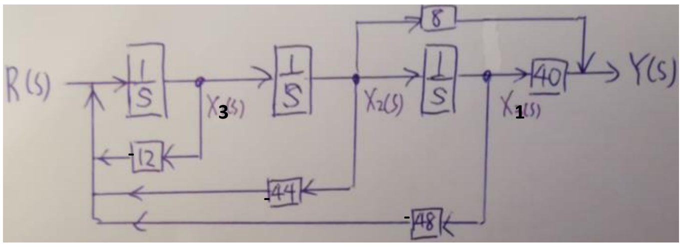
$$T(s) = \frac{Y(s)}{R(s)} = \frac{8(s+5)}{s^3 + 12s^2 + 44s + 48}$$

(a) Determine a state variable model.

(b) Determine $\mathbf{\Phi}(t)$, the state transition matrix.

Ans.

(a) $\frac{Y(s)}{R(s)} = \frac{8(s+5)}{s^3+12s^2+44s+48} = \frac{8s^{-2}+40s^{-3}}{1+12^{-1}+44s^{-2}+48s^{-3}}$



$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathbf{r}, \mathbf{y} = \begin{bmatrix} 40 & 8 & 0 \end{bmatrix}\mathbf{x}$$

(b) $\mathbf{\Phi}(t) = e^{\mathbf{A}t} = [\Phi_1(t) \quad \Phi_2(t) \quad \Phi_3(t)]$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -48 & -44 & -12 \end{bmatrix} \mathbf{\Phi}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} \Rightarrow \mathbf{\Phi}(t)$$

$$\Phi_1(t) = \begin{bmatrix} e^{-6t} - 3e^{-4t} + 3e^{-2t} \\ -6e^{-6t} + 12e^{-4t} - 6e^{-2t} \\ 36e^{-6t} - 48e^{-4t} + 12e^{-2t} \end{bmatrix} \quad \Phi_2(t) = \begin{bmatrix} \frac{3}{4}e^{-6t} - 2e^{-4t} + \frac{5}{4}e^{-2t} \\ -\frac{9}{2}e^{-6t} + 8e^{-4t} - \frac{5}{2}e^{-2t} \\ 27e^{-6t} - 32e^{-4t} + 5e^{-2t} \end{bmatrix} \Phi_3(t) = \begin{bmatrix} \frac{1}{8}e^{-6t} - \frac{1}{4}e^{-4t} + \frac{1}{8}e^{-2t} \\ -\frac{3}{4}e^{-6t} + e^{-4t} - \frac{1}{4}e^{-2t} \\ \frac{9}{2}e^{-6t} - 4e^{-4t} + \frac{1}{2}e^{-2t} \end{bmatrix}$$