# Chapter 9

Stability in Frequency Domain

# 9.1 Introduction

S domain

- Stability and relative stability

→Routh-Hurwitz criterion

→Root locus

- Terminologies related to design specs

→ Damping ratio, natural frequency

Frequency domain

- Stability and relative stability

→Nyquist stability criterion

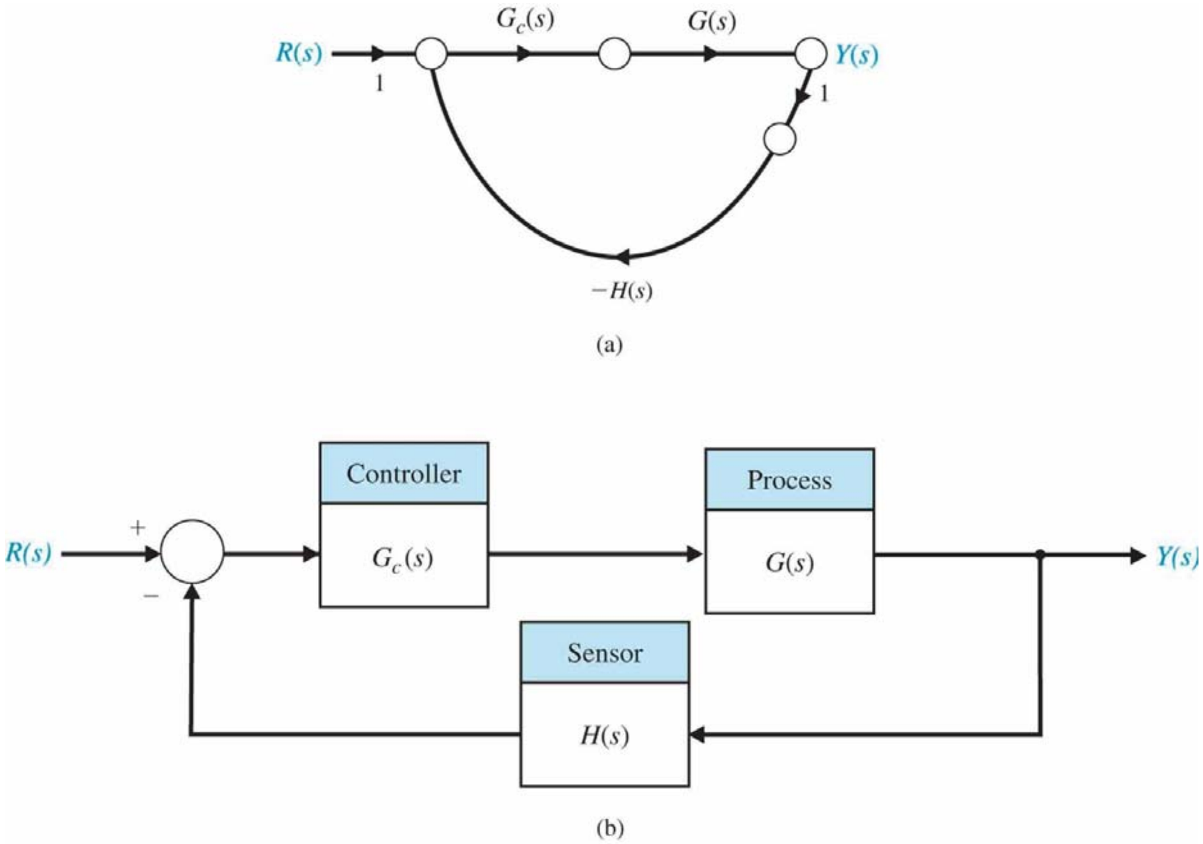
→Bode plot

- Terminologies related to design specs

→ Gain margin, phase margin, bandwidth

Work on characteristic equation in the following form:

$$1+L(s)=0$$




Note: For multiloop systems, char. eq. can still be expressed as 1+L(s)=0

# 9.2 Mapping Contour in s-Plane

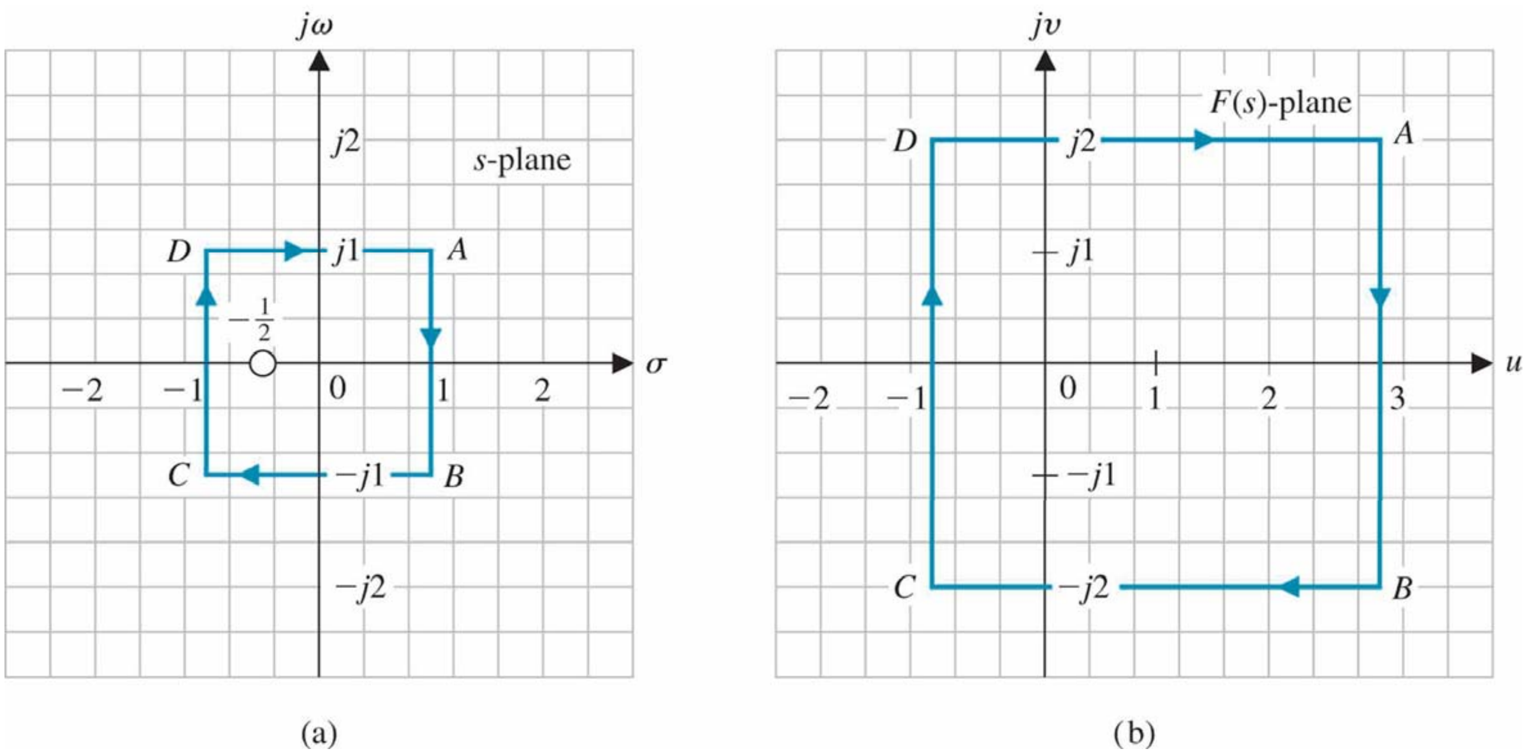



**Figure 9.2** Mapping a square contour by $F(s) = 2s + 1 = 2(s + 1/2)$.

**Contour map:** A contour/trajectory in one plane is mapped/translated into another plane by a relation F(s).

**Positive contour:** clockwise traversal of a contour.

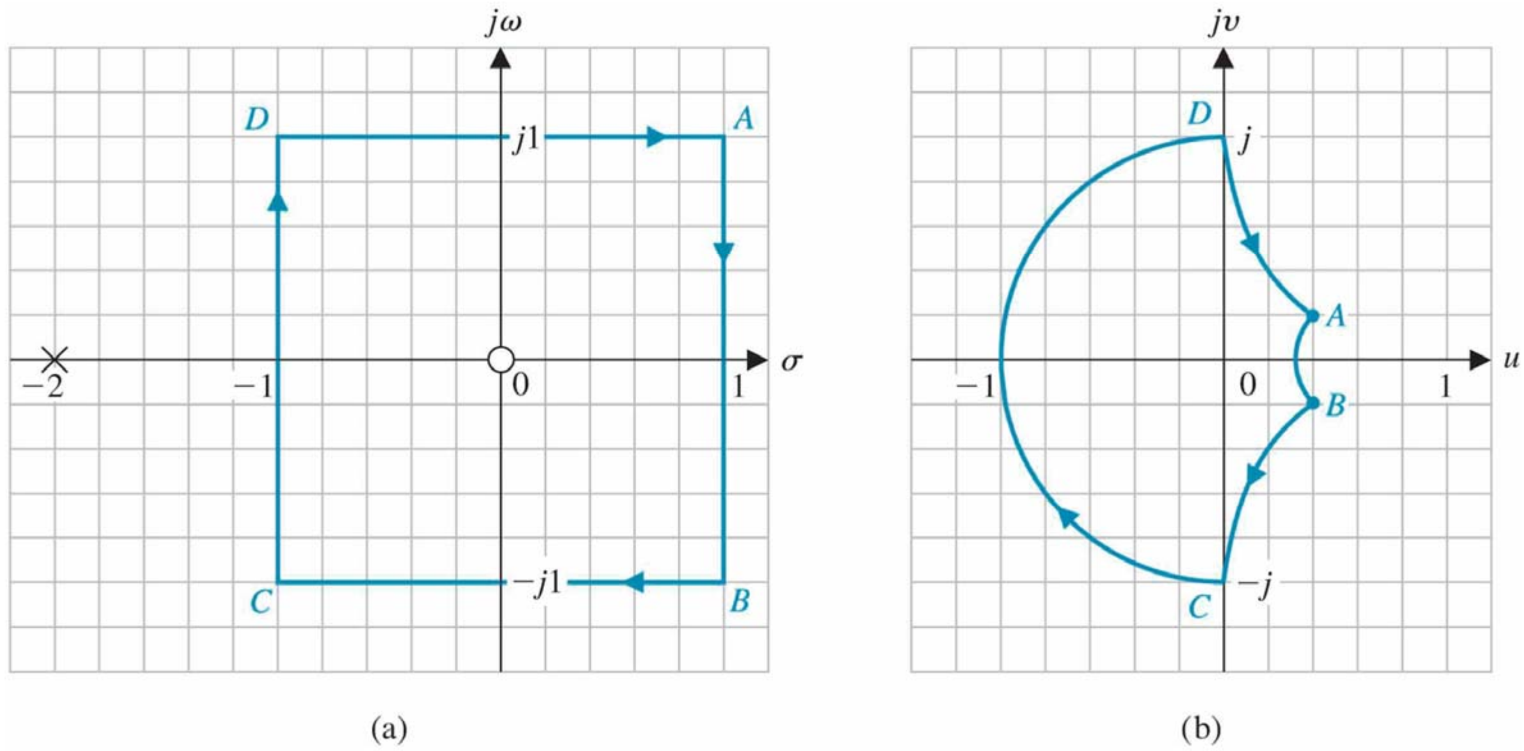



Figure 9.3 Mapping for $F(s) = s/(s + 2)$.

- Typically, we are concerned with an F(s) that is a rational function of s
- Area enclosed by a contour: the area within a contour to the right of the traversal of the contour

Principle of the argument (Cauchy's theorem):
If a positive contour in the s-plane encircles Z zeros and P poles of F(s) and does not pass through any poles or zeros of F(s), then the corresponding contour in the F(s)-plane positively encircles the origin N=Z-P times.

Note (See the derivation related to (9.11) in the textbook):
1. N<0 means negatively encirclement.
2. In the F(s)-plane, if the origin is ``on'' the contour, then it is not considered as being encircled.

Chiu's Reminiscence:

Bode diagram:

Pole→-20dB/decade

Zero→+20dB/decade

Nyquist diagram:

Pole→negative encirclement of the origin

Zero→positive encirclement of the origin

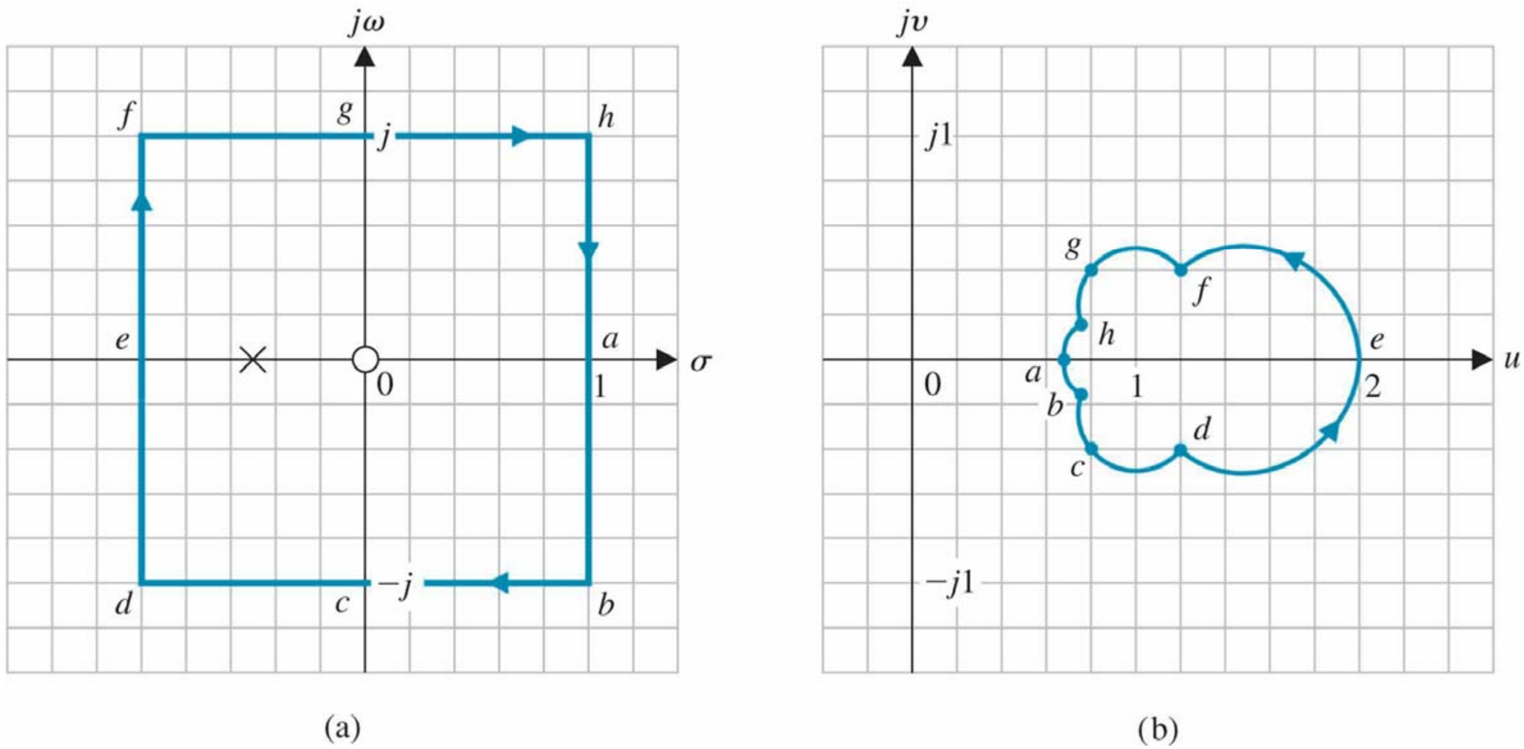



Figure 9.4 Mapping for $F(s) = s/(s + 1/2)$.

## 9.2 Mapping Contours in the *s*-PLANE

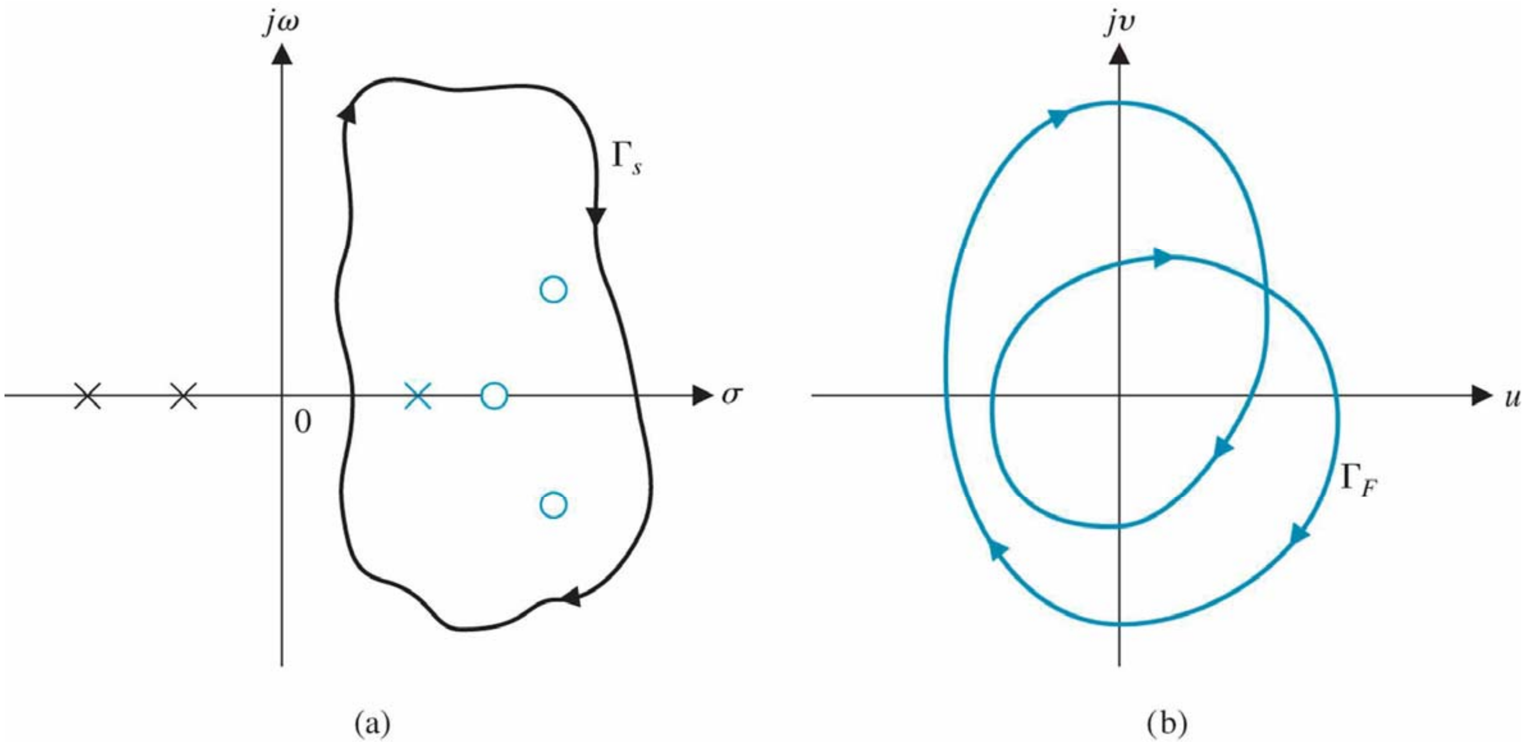



Figure 9.6 Example of Cauchy's theorem with three zeros and one pole within $\Gamma_s$.

## 9.2 Mapping Contours in the *s*-PLANE

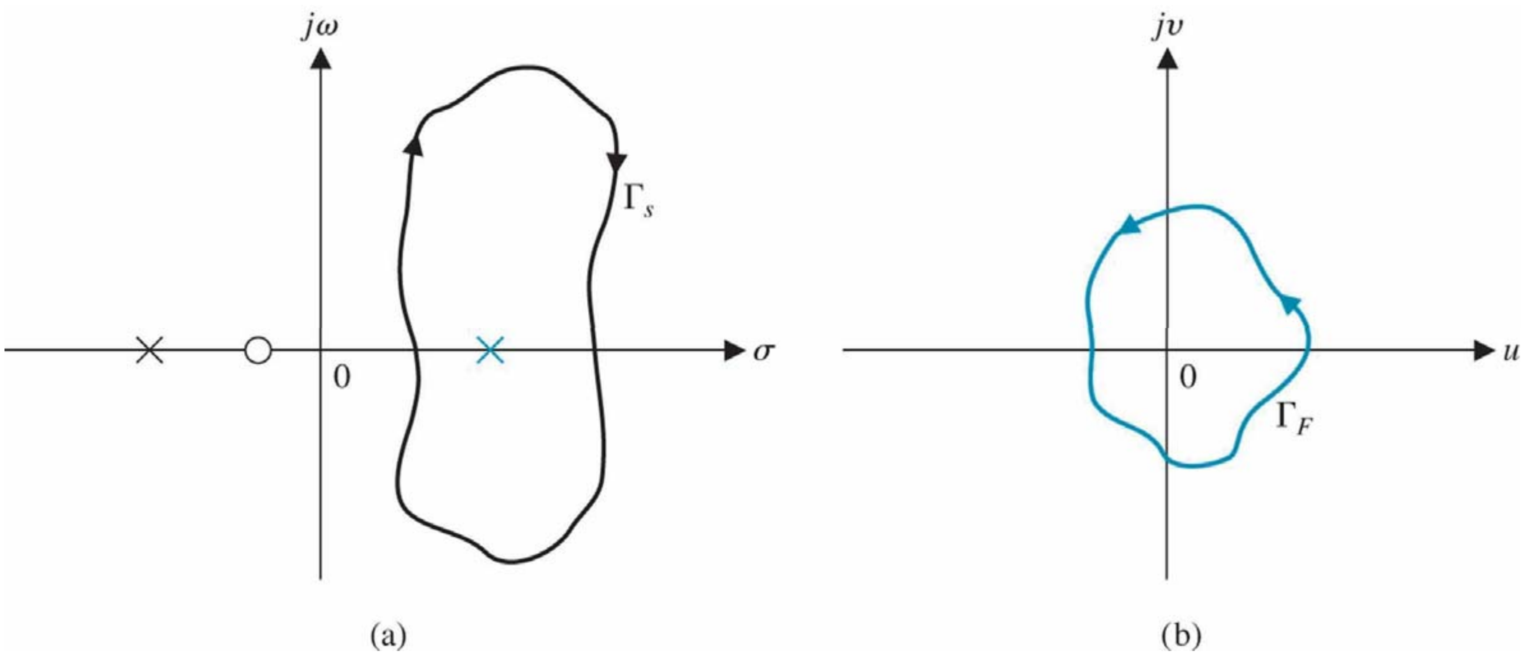



Figure 9.7 Example of Cauchy's theorem with one pole within $\Gamma_s$.

# 9.3 The Nyquist Criterion

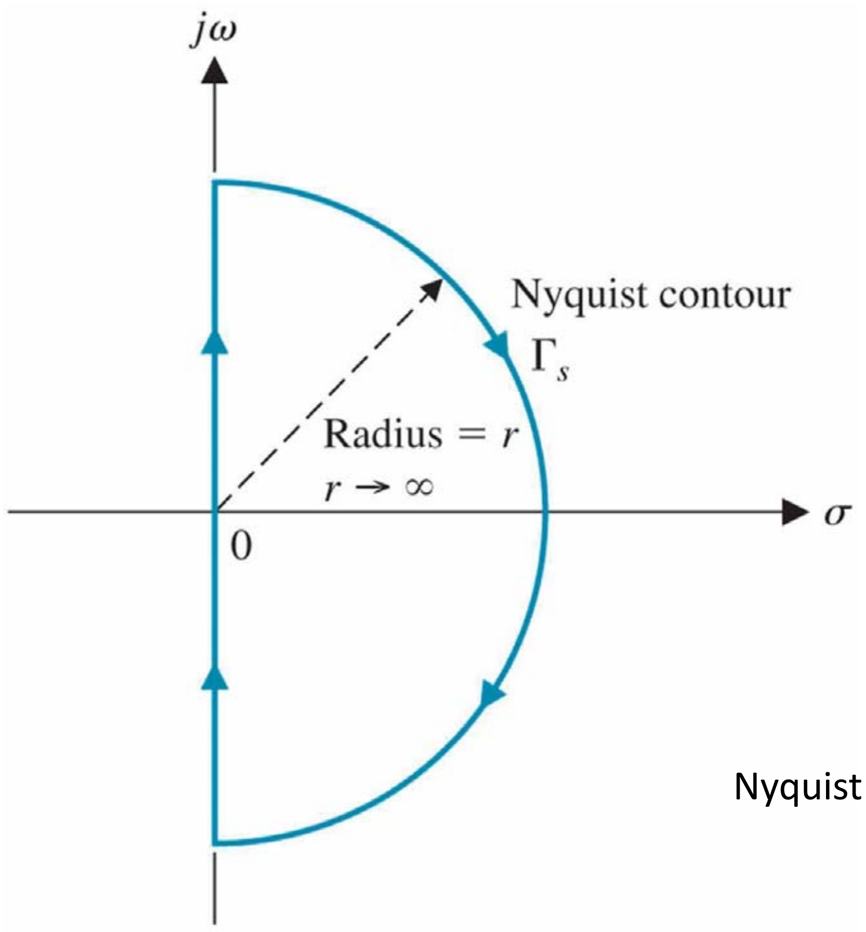



Figure 9.8 Nyquist contour is shown as the heavy line.

Nyquist plot: Polar plot using the Nyquist contour

Consider **Nyquist** plots of $F(s)=1+L(s)$ and $L(s)$;
→ $N_F=Z_F-P_F$ and $N_L=Z_L-P_L$

- $Z_F$ (to be determined)

= # zeros of $F(s)$ in the right-half s-plane
= # poles of the closed-loop transfer function in the right-half s-plane
= # roots of the characteristic equation in the right-half s-plane
→ Unstable if $N_L>0$

- $N_F$

= # positive encirclement of (0,0) from **Nyquist** plot of $F(s)$
= # positive encirclement of (-1,0) from **Nyquist** plot of $L(s)$

- $P_F$

= $P_L$ (poles of $F(s)$=poles of $L(s)$)

# Nyquist stability criterion

- # positive encirclement of (-1,0) from **Nyquist** plot of L(s)=$Z_F$-$P_L$

Note:

1. For a stable loop transfer function, the closed-loop system is stable if **Nyquist** plot of L(s) does not encircle point (-1,0) or pass through that point.

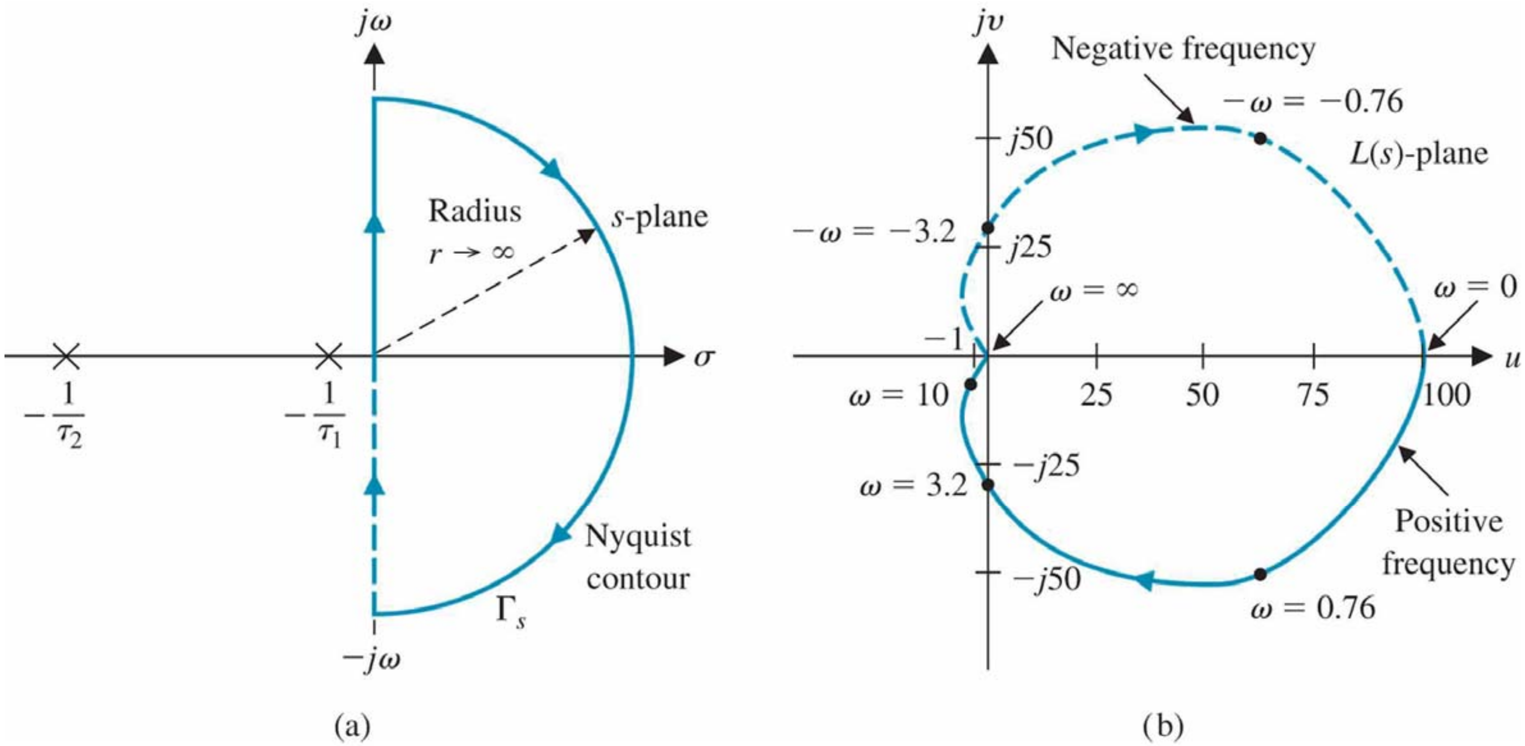
   **A feedback system is stable if and only if the contour $\Gamma_L$ in the $L(s)$-plane does not encircle the $(-1, 0)$ point when the number of poles of $L(s)$ in the right-hand $s$-plane is zero $(P = 0)$.**

2. For an unstable loop transfer function:

   **A feedback control system is stable if and only if, for the contour $\Gamma_L$, the number of counterclockwise encirclements of the $(-1, 0)$ point is equal to the number of poles of $L(s)$ with positive real parts.**

3. A root is on jw if Nyquist plot passes through point (-1,0).

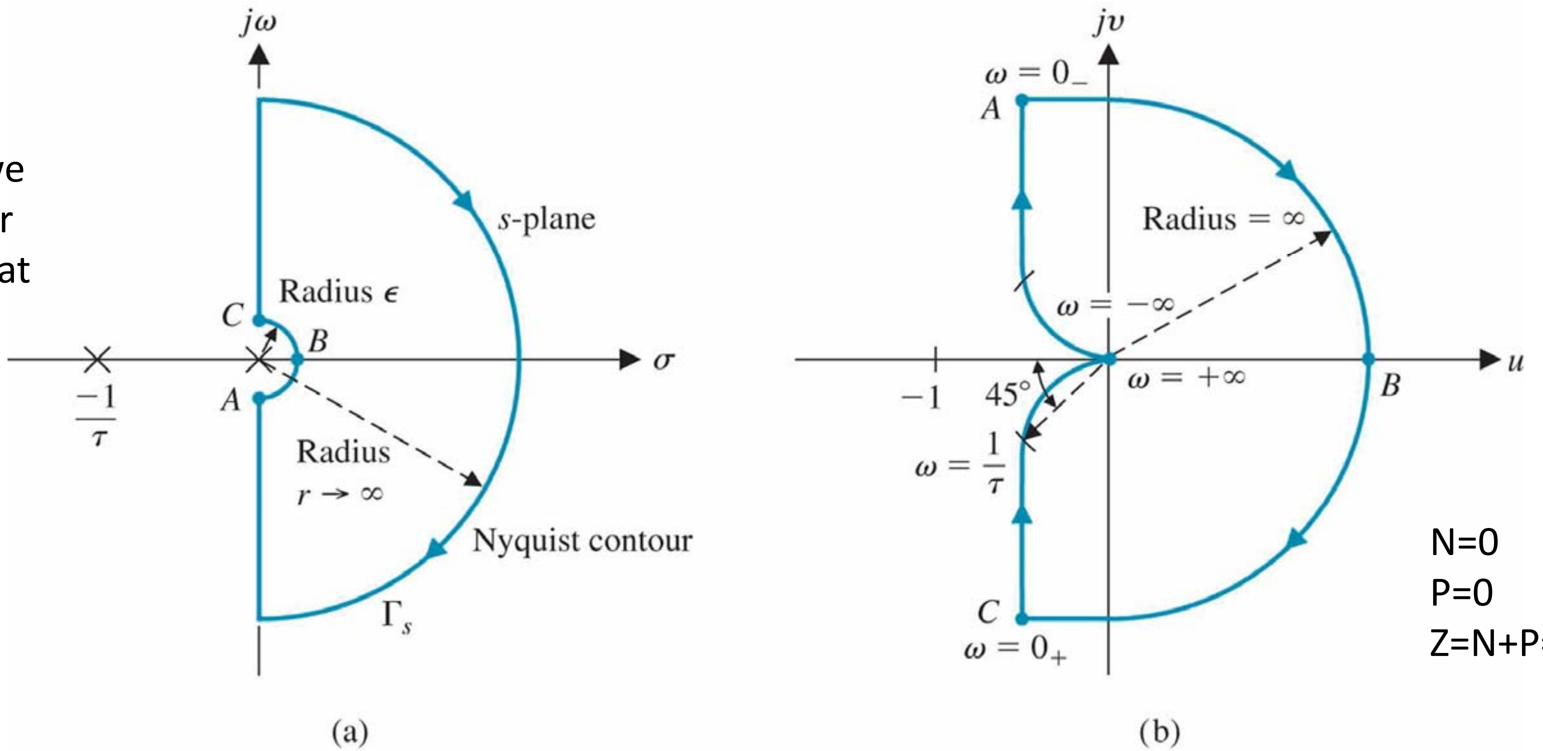
Polar plot




N=0
P=0
Z=N+P=0 (stable)

Figure 9.9 Nyquist contour and mapping for $L(s) = \frac{100}{(s+1)(s/10+1)}$.

By convention, we consider a detour around the pole at the origin




N=0
P=0
Z=N+P=0 (stable)

Figure 9.10 Nyquist contour and mapping for $L(s) = K/(s(\tau s + 1))$.

For what value of K is the system stable?

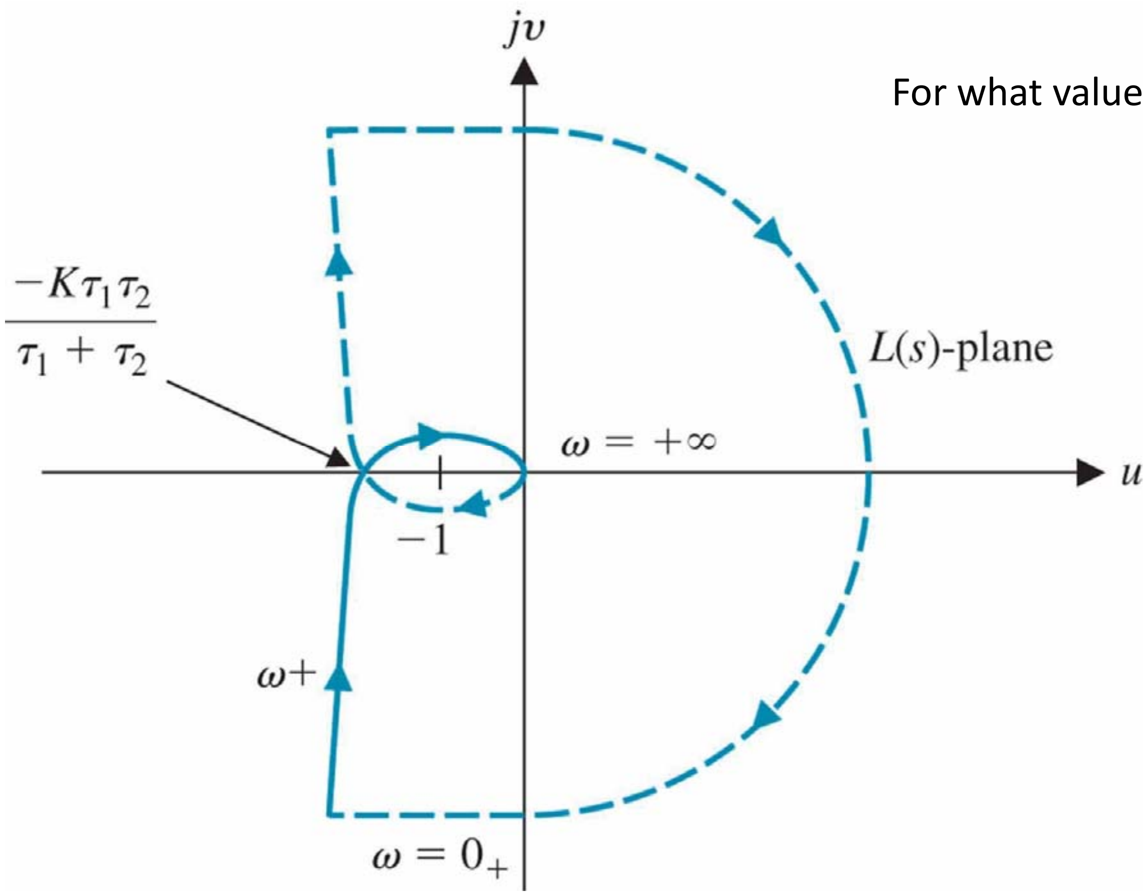



Figure 9.11 Nyquist diagram for $L(s) = K/(s(\tau_1 s + 1)(\tau_2 s + 1))$. The tic mark shown to the left of the origin is the −1 point.

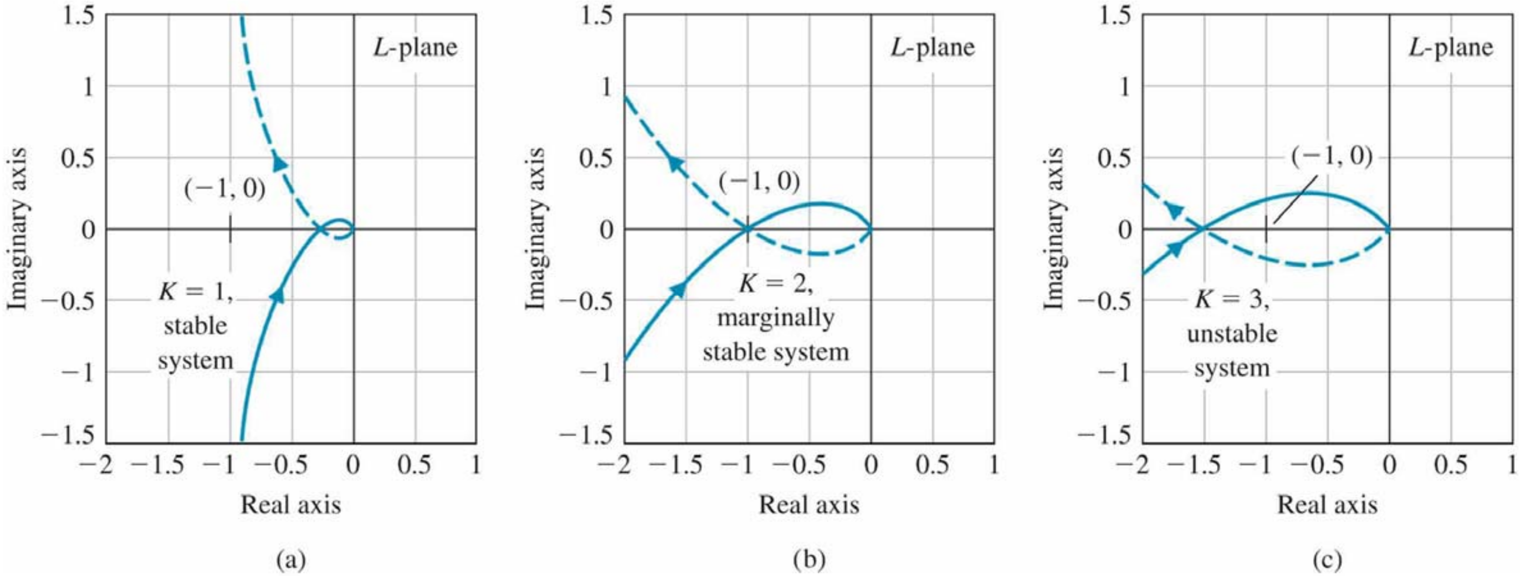
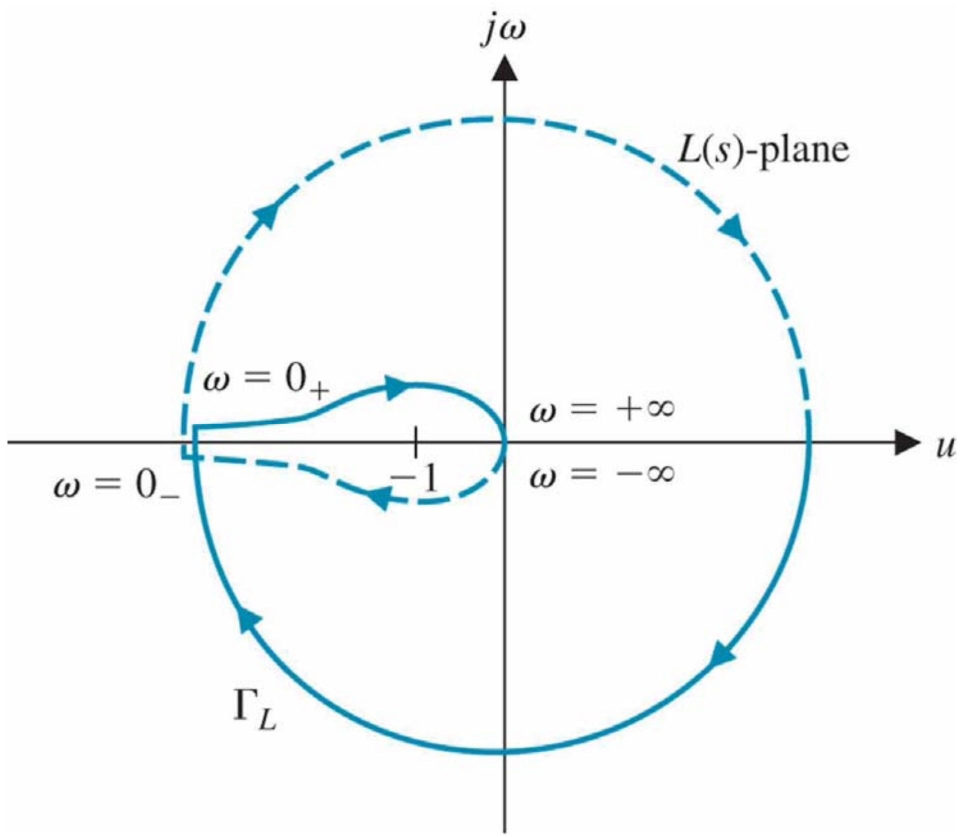



Figure 9.12 Nyquist plot for $L(s) = G_c(s)G(s)H(s) = \dfrac{K}{s(s+1)^2}$ when (a) $K = 1$, (b) $K = 2$, and (c) $K = 3$.




N=2
P=0
Z=N+P=2 (unstable)

Figure 9.13 Nyquist contour plot for $L(s) = K/(s^2(\tau s + 1))$.

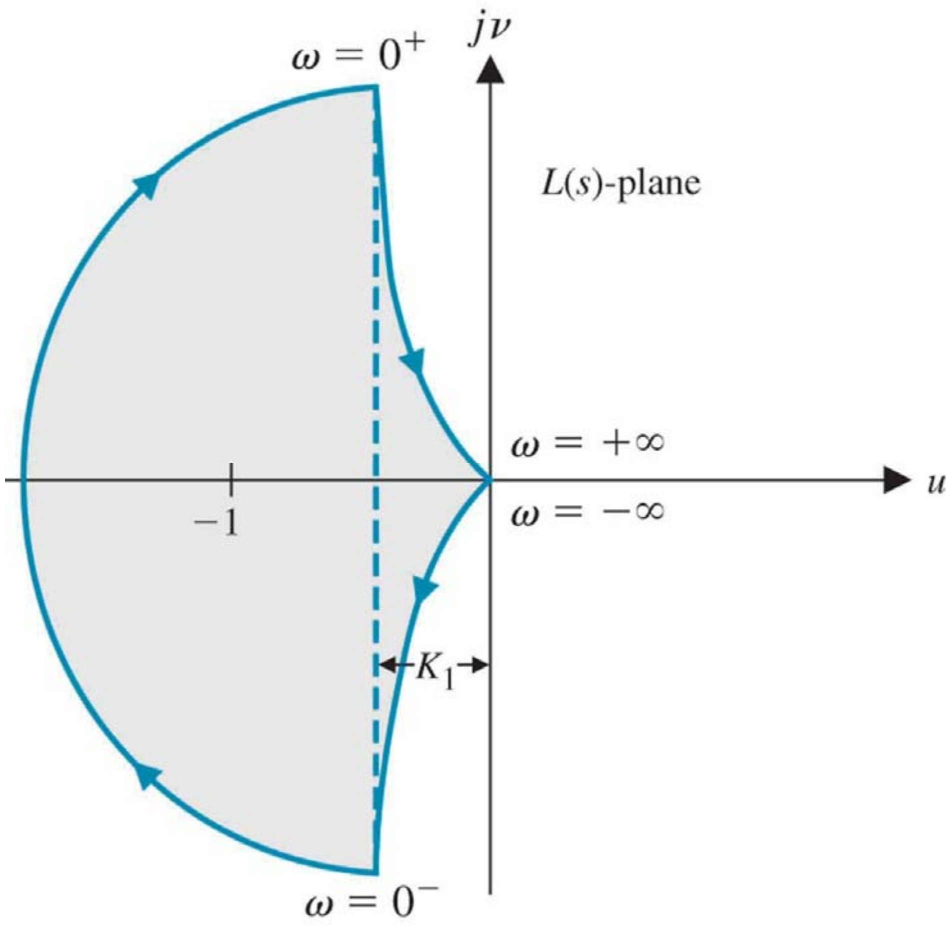
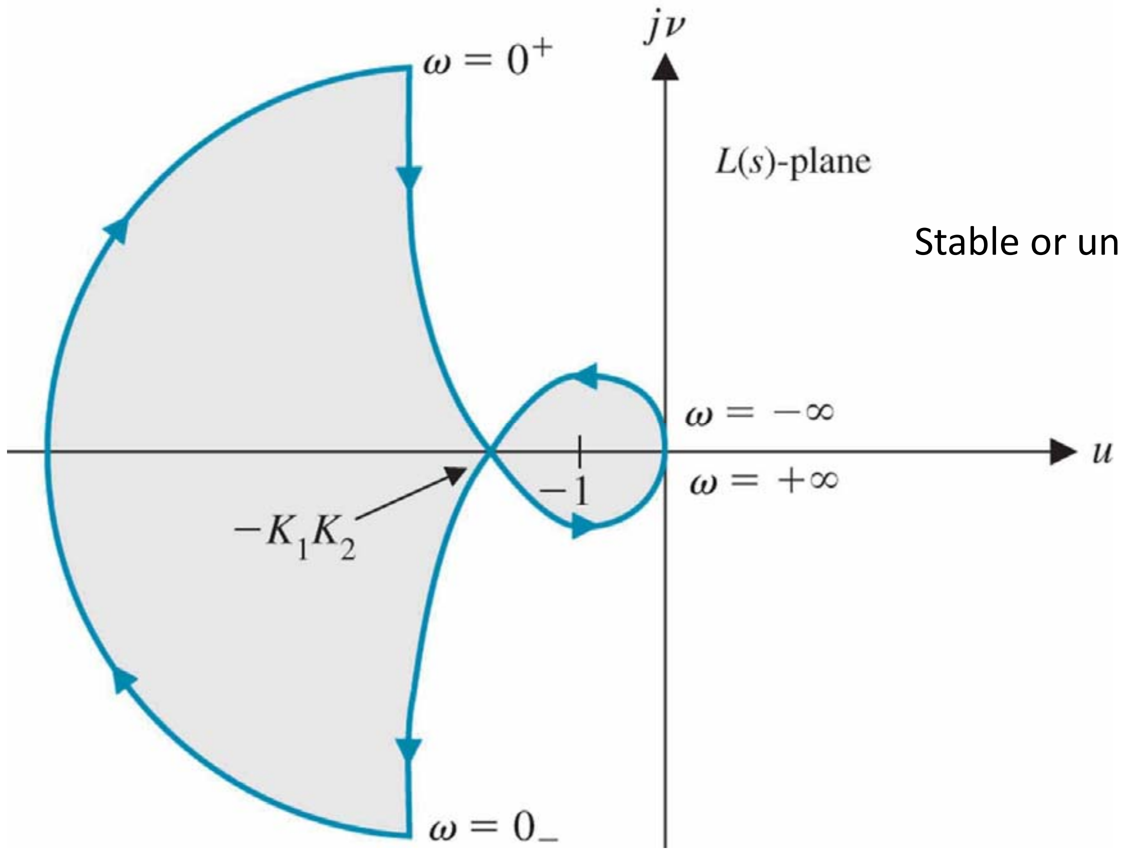



**Figure 9.15** Nyquist diagram for $L(s) = K_1/(s(s - 1))$.

N=1
P=1
Z=N+P=2 (unstable)




Stable or unstable?

P=1

- $-K_1K_2<-1$
  N=-1
  Z=N+P=0 (stable)
- $-K_1K_2=-1$
  N=0
  Z=N+P=1 (unstable)
- $-K_1K_2>-1$
  N=1
  Z=N+P=2 (unstable)

Figure 9.16 Nyquist diagram for $L(s) = K_1(1 + K_2s)/(s(s - 1))$.

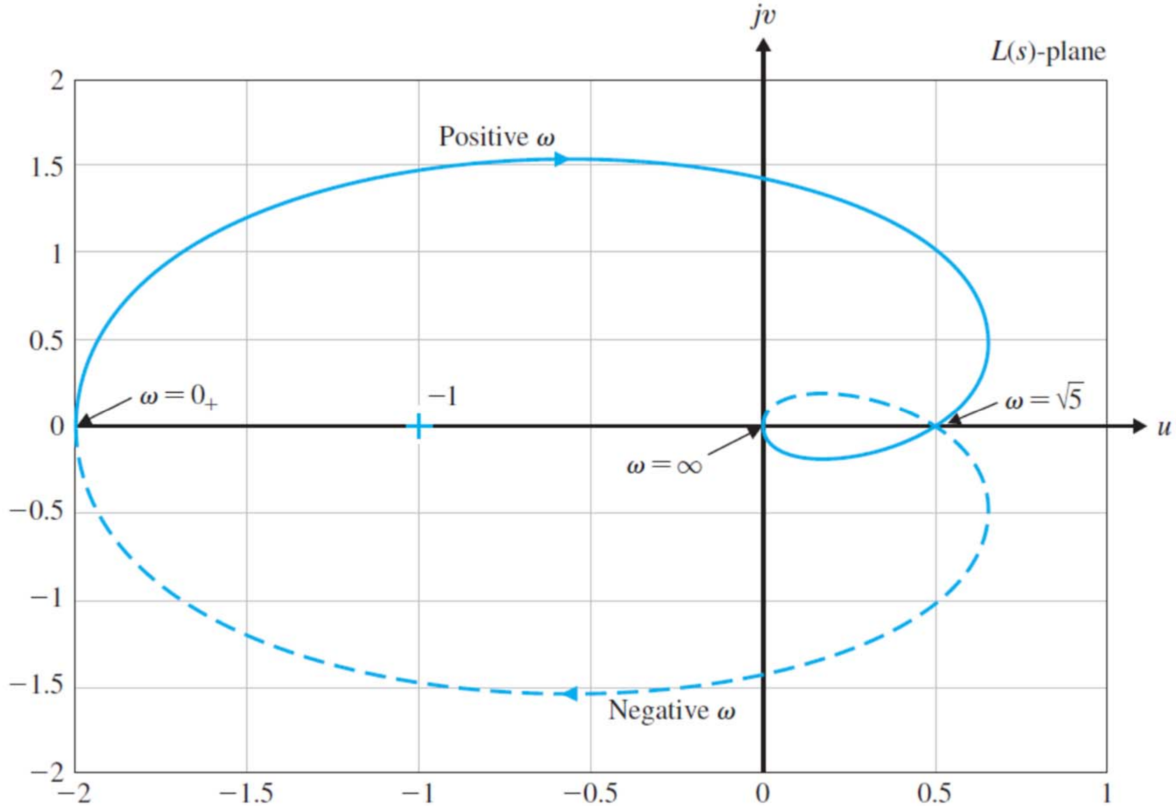



**FIGURE 9.17** Nyquist plot for Example 9.6 for $L(j\omega)/K$.

$$L(s) = G_c(s)G(s) = \frac{K(s-2)}{(s+1)^2}.$$

P=0

- -2K<-1
  N=1
  Z=N+P=1 (unstable)
- -2K>=-1
  N=0
  Z=N+P=0 (stable)

# 9.4 Relative Stability and the Nyquist Criterion

- For the s-plane, we defined the relative stability of a system as the property measured by the relative settling time of each root or pair of roots.

→ $T_s = 4\tau$, which is related to the real parts of the roots

→System with a shorter settling time is considered relatively more stable

- We would like to determine a similar measure of relative stability useful for the frequency response method.

# Gain Margin

$$L(j\omega) = G_c(j\omega)G(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

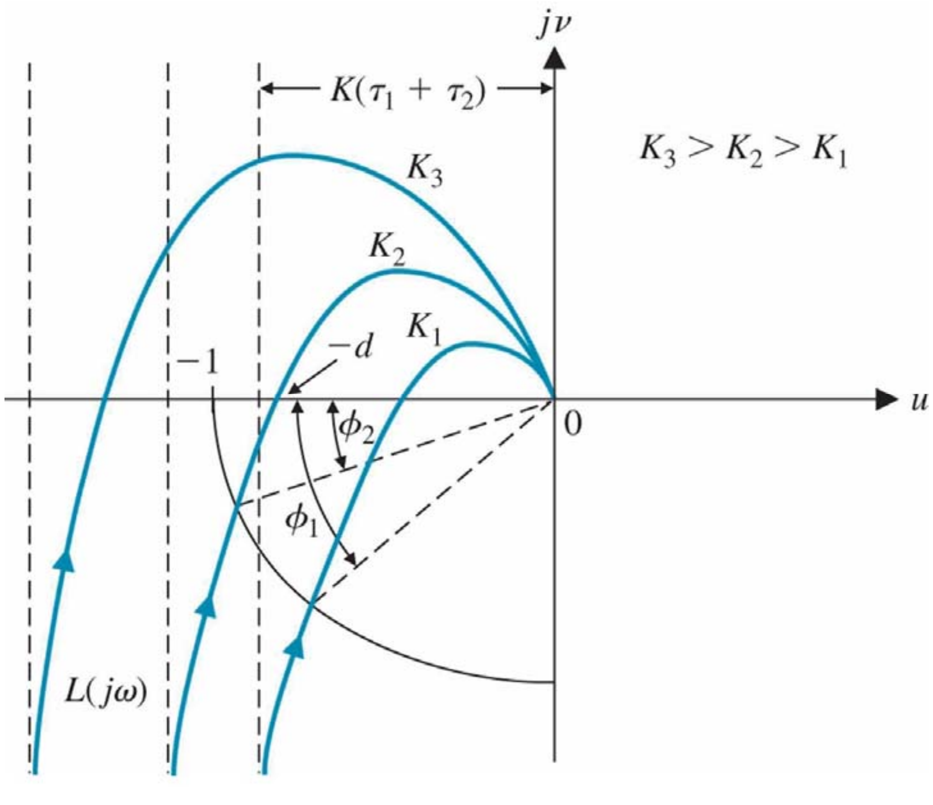



Figure 9.18 Polar plot for $L(j\omega)$ for three values of gain.

- Gain margin

For stable L(s), gain margin is the additional gain that can be added before the system becomes unstable

GM:= 20log 1/|L(jw$_{pc}$)|, where phase crossover frequency $w_{pc}$ is the frequency that makes ∠L(jw$_{pc}$)=-180$^{o}$

Why? hint: 0 db – 20log |L(jw)|

# Phase Margin

$$L(j\omega) = G_c(j\omega)G(j\omega) = \frac{K}{j\omega(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)}$$

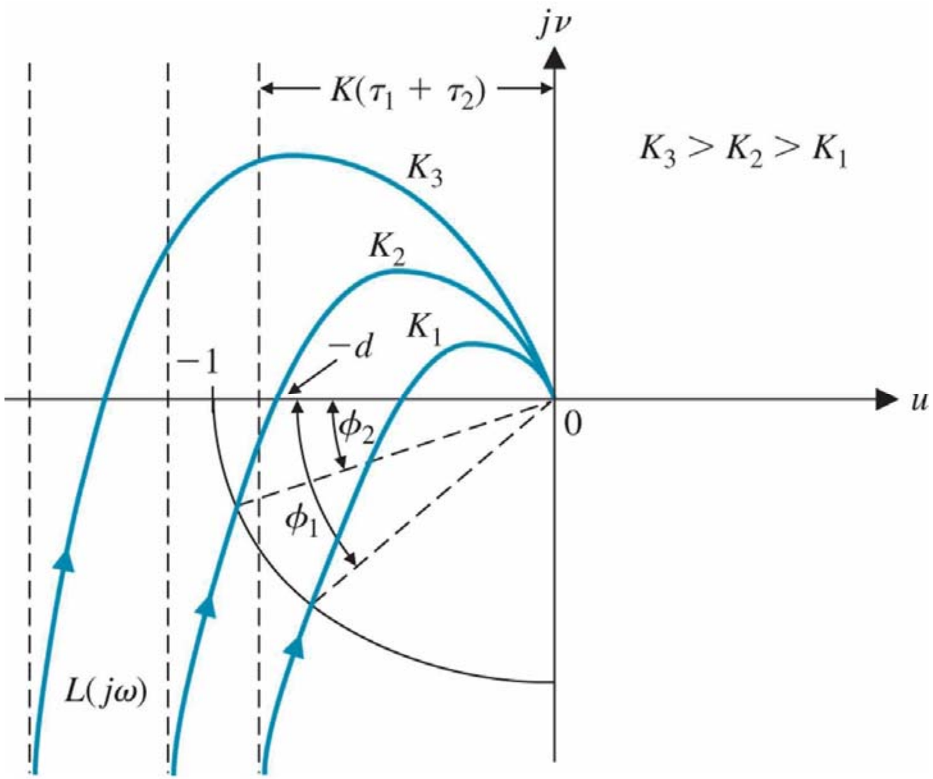



Figure 9.18 Polar plot for $L(j\omega)$ for three values of gain.

- Phase margin

For stable L(s), phase margin is the additional phase lag required before the system becomes unstable ($-180^o$)

For system with gain $K_1$, PM=$\phi_1$

For system with gain $K_2$, PM=$\phi_2$

Gain crossover frequency: the frequency $w_{gc}$ that makes $|L(jw_{gc})|$=0 dB

# Margins and Crossover Frequencies

- Gain crossover frequency $w_{gc}$

→The frequency that makes loop gain 0 dB

- Phase crossover frequency $w_{pc}$

→The frequency that makes loop phase $-180^o$

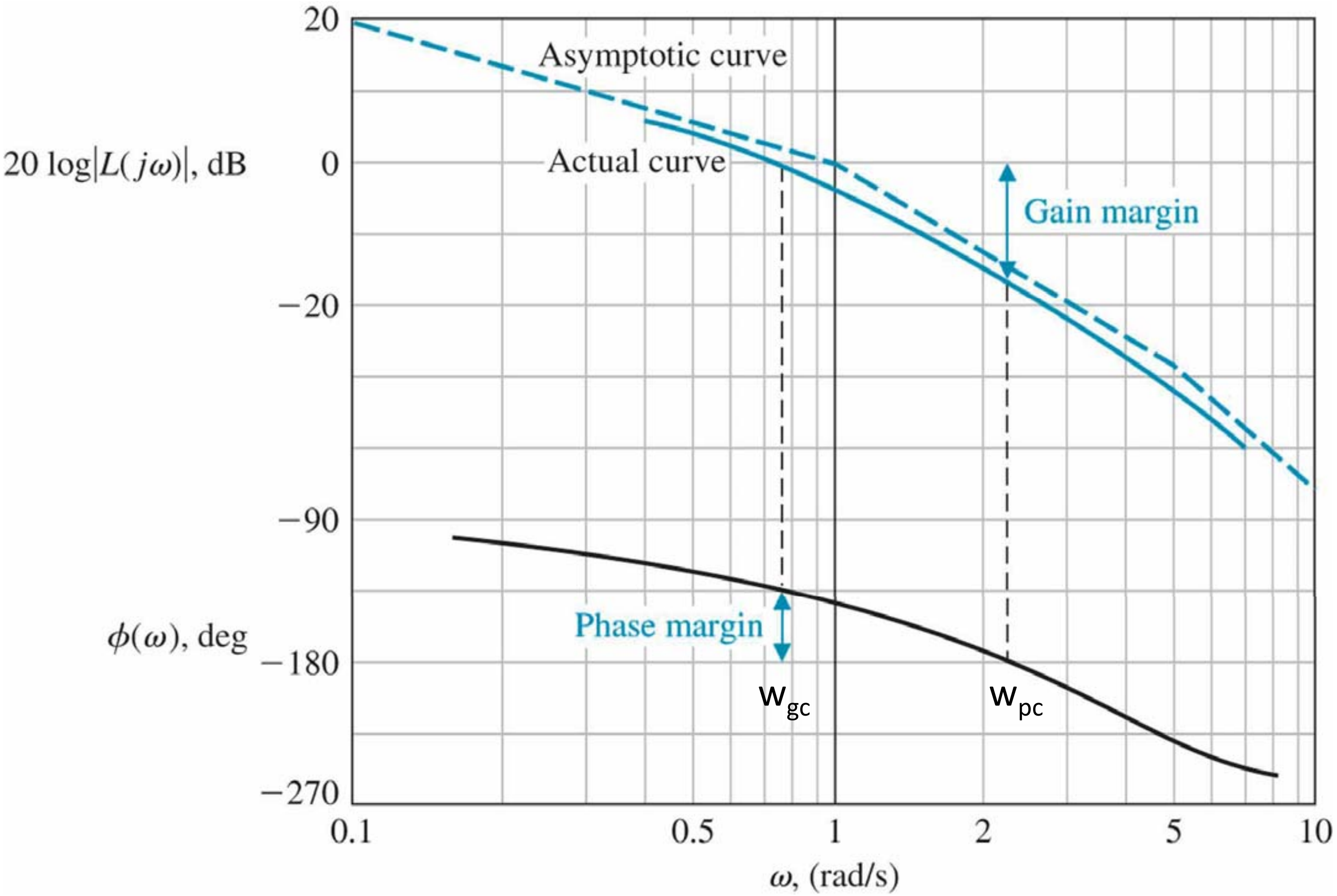
- Gain margin=$20\log(1/|L(jw_{pc})|)$

→Additional gain to be added before system becomes unstable

- Phase margin= $\angle L(jw_{gc})-(-180^o)$

→Additional phase lag required before the system becomes unstable

# GM and PM in Bode Plot




$w_{gc}$ gain crossover frequency

$w_{pc}$ phase crossover frequency

Figure 9.19 Bode diagram for $L(j\omega) = 1/(j\omega(j\omega + 1)\ (0.2\ j\omega + 1))$.

# GM and PM in Log-Magnitude–Phase Plot

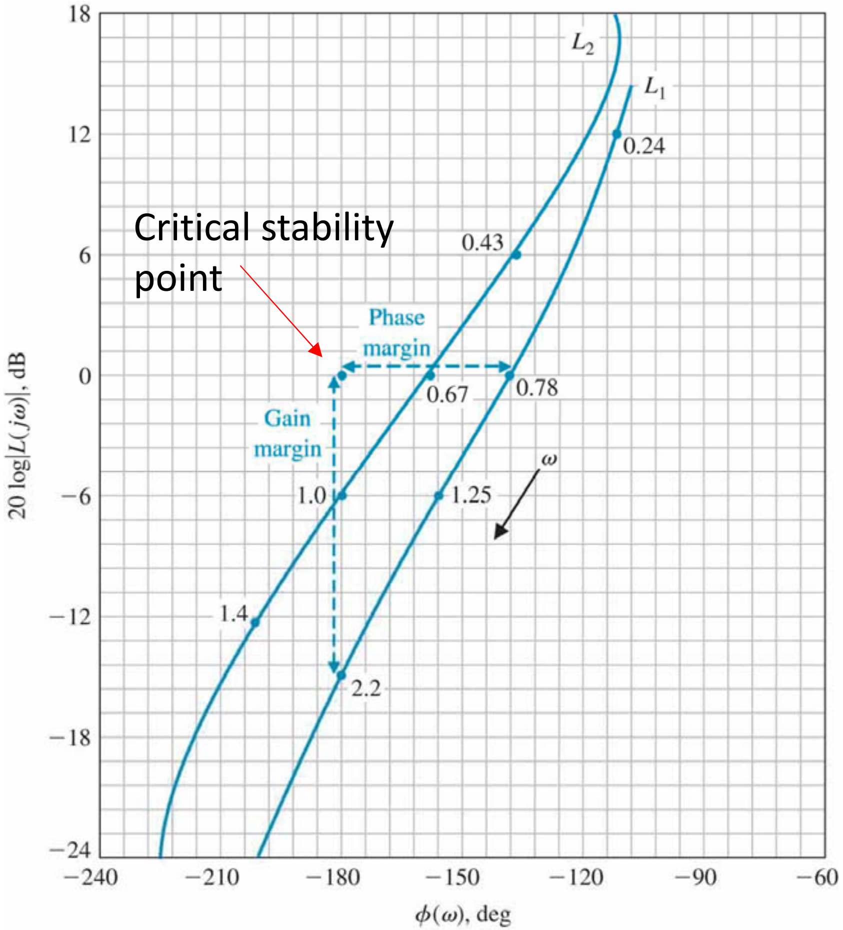



Figure 9.20 Log-magnitude– phase curve for $L_1$ and $L_2$.

L1: Gain margin=15 dB
Phase magin=43°
L2: Gain margin=5.7 dB
Phase magin=20°

Feedback system of $L_2$ is relatively less table than feedback system of $L_1$

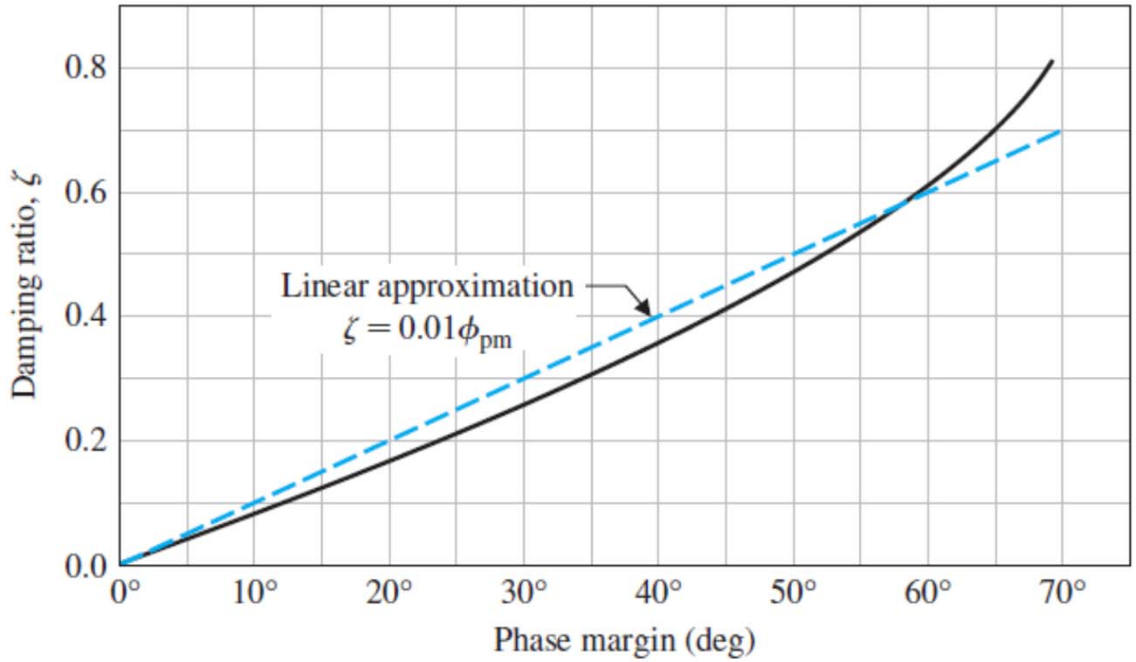
What are the gain and phase crossover frequencies?

# Damping Ratio and PM for 2nd-order System

- Loop TF $$L(s) = G_c(s)G(s) = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}.$$
- Sinusoidal steady-state TF $$L(j\omega) = \frac{\omega_n^2}{j\omega(j\omega + 2\zeta\omega_n)}.$$
- At gain crossover frequency $$\frac{{\omega_n}^2}{\omega_c({\omega_c}^2 + 4\zeta^2{\omega_n}^2)^{1/2}} = 1 \quad \Rightarrow \quad \frac{{\omega_c}^2}{{\omega_n}^2} = (4\zeta^4 + 1)^{1/2} - 2\zeta^2.$$
- PM

$$\begin{aligned}\phi_{\text{pm}} &= 180^\circ - 90^\circ - \tan^{-1}\frac{\omega_c}{2\zeta\omega_n} \\ &= 90^\circ - \tan^{-1}\left(\frac{1}{2\zeta}[(4\zeta^4 + 1)^{1/2} - 2\zeta^2]^{1/2}\right) \\ &= \tan^{-1}\frac{2}{[(4 + 1/\zeta^4)^{1/2} - 2]^{1/2}}.\end{aligned}$$

$\Rightarrow$ $\boxed{\zeta = 0.01\phi_{\text{pm}},}$ $\quad \zeta \leq 0.7$




- $\zeta = 0.01\phi_{pm}$, a suitable approximation for a second-order system and may be used for higher-order systems if the transient response of the system is primarily due to a pair of dominant underdamped roots.
- The phase margin and the gain margin are suitable measures of the performance of the system.
- We normally emphasize phase margin as a frequency- domain specification.

# 9.5 Time-Domain Performance Criteria in the Frequency Domain

- Transient performance of a feedback system can be estimated from the closed-loop frequency response

→Resonant peak is related to damping ratio

$$M_{p\omega} = |T(\omega_r)| = (2\zeta\sqrt{1-\zeta^2})^{-1}, \qquad \zeta < 0.707.$$

- The open- and closed-loop frequency responses for a single-loop system are related

→open-loop TF is used to analyze the properties of closed-loop TF, e.g., Nyquist criterion and the phase margin index

Why?

- Because this relationship between the closed-loop frequency response and the transient response is a useful one, we would like to be able to determine resonant peak from the Nyquist plots

# Constant M circles

## What?

- M-circles can determine the closed-loop magnitude response from open-loop response

## How?

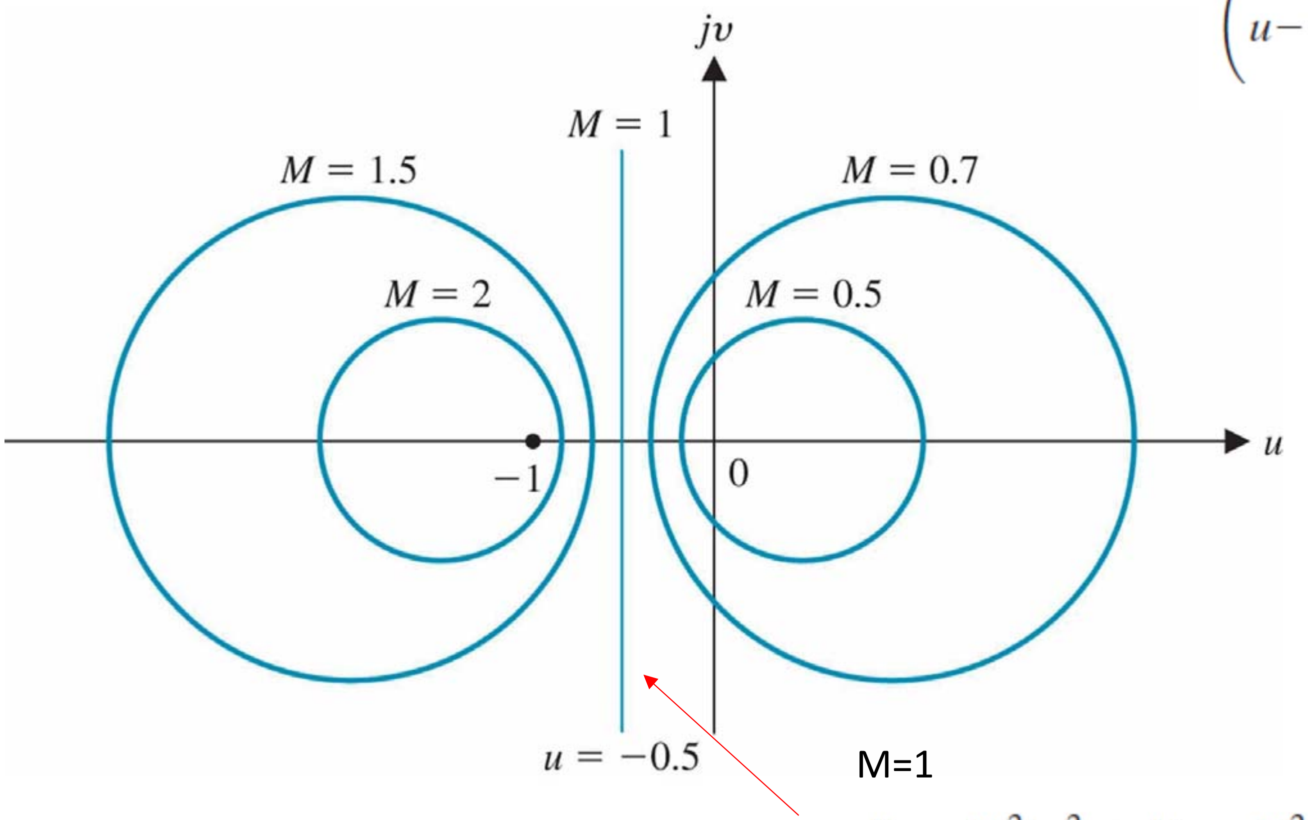
Open loop $L(j\omega) = G_c(j\omega)G(j\omega) = u + jv.$

Closed loop $M(\omega) = \left|\frac{G_c(j\omega)G(j\omega)}{1 + G_c(j\omega)G(j\omega)}\right| = \left|\frac{u + jv}{1 + u + jv}\right| = \frac{(u^2 + v^2)^{1/2}}{[(1 + u)^2 + v^2]^{1/2}}.$

➡ $(1 - M^2)u^2 + (1 - M^2)v^2 - 2M^2u = M^2.$

➡ $\left(u - \frac{M^2}{1 - M^2}\right)^2 + v^2 = \left(\frac{M}{1 - M^2}\right)^2,$

$$\left(u - \frac{M^2}{1 - M^2}\right)^2 + v^2 = \left(\frac{M}{1 - M^2}\right)^2,$$




M=1

$$(1 - M^2)u^2 + (1 - M^2)v^2 - 2M^2u = M^2.$$

Figure 9.23 Constant *M* circles.

Resonant peak and frequency

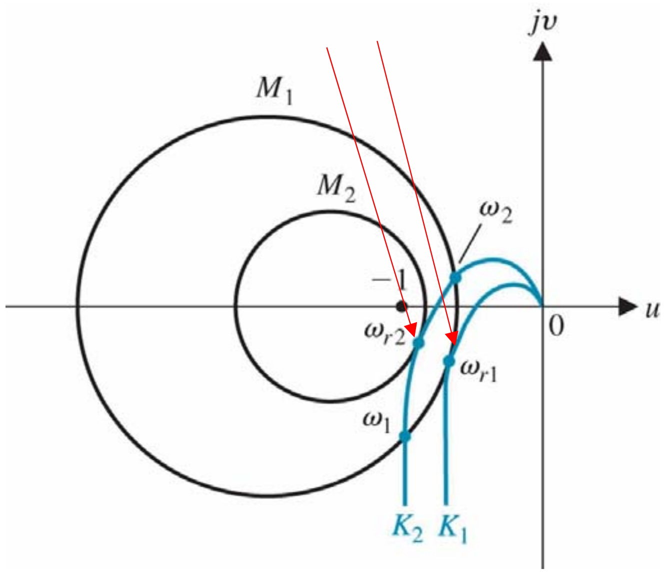



Figure 9.24 Polar plot of $G_c(j\omega)G(j\omega)$ for two values of a gain ($K_2 > K_1$).

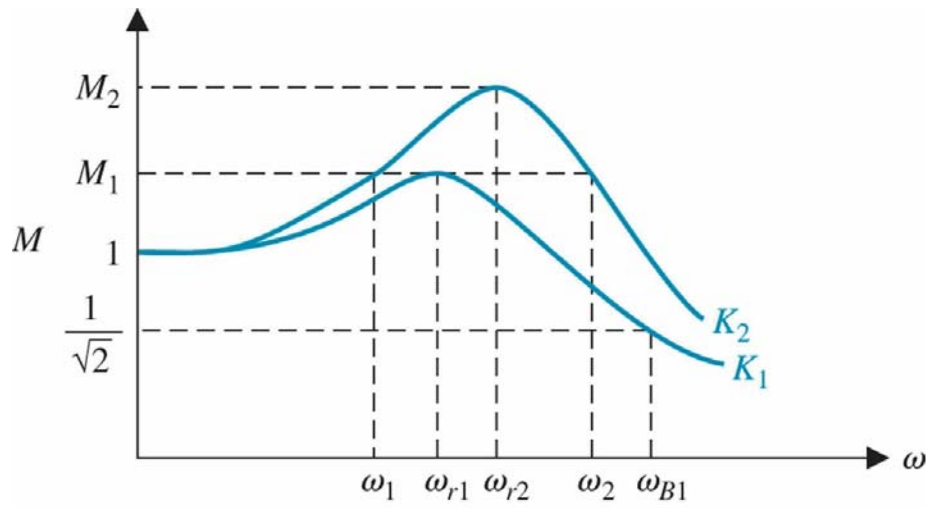



Figure 9.25 Closed-loop frequency response of $T(j\omega) = G_c(j\omega)G(j\omega)/(1 + G_c(j\omega)G(j\omega))$. Note that $K_2 > K_1$.

# Constant N circles

- Constant N circles relate the open-loop Nyquist plot to the angles of the closed–loop system

$$\phi = \underline{/T(j\omega)} = \underline{/(u + jv)/(1 + u + jv)}$$

$$= \tan^{-1}\left(\frac{v}{u}\right) - \tan^{-1}\left(\frac{v}{1+u}\right).$$

$$\Rightarrow \quad u^2 + v^2 + u - \frac{v}{N} = 0, \qquad N = \tan\phi.$$

$$\Rightarrow \quad \left(u + \frac{1}{2}\right)^2 + \left(v - \frac{1}{2N}\right)^2 = \frac{1}{4}\left(1 + \frac{1}{N^2}\right),$$

# Nichols Chart (Log-magnitude–phase diagram (+M and N circles)

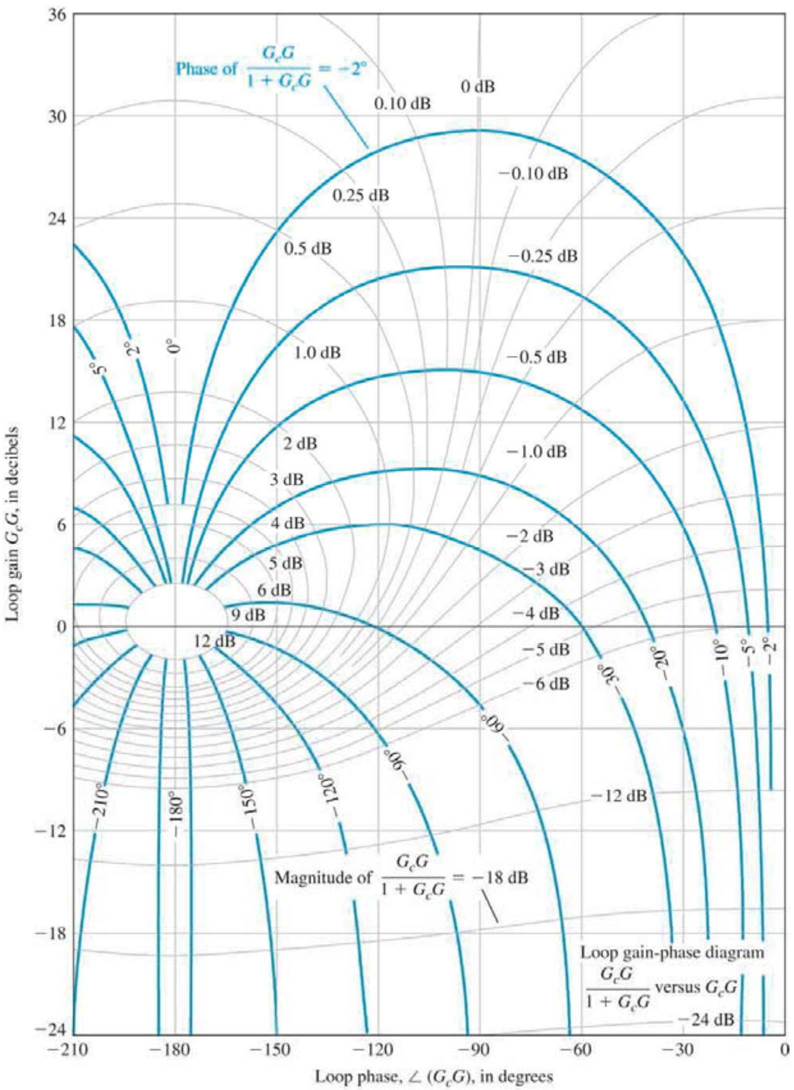
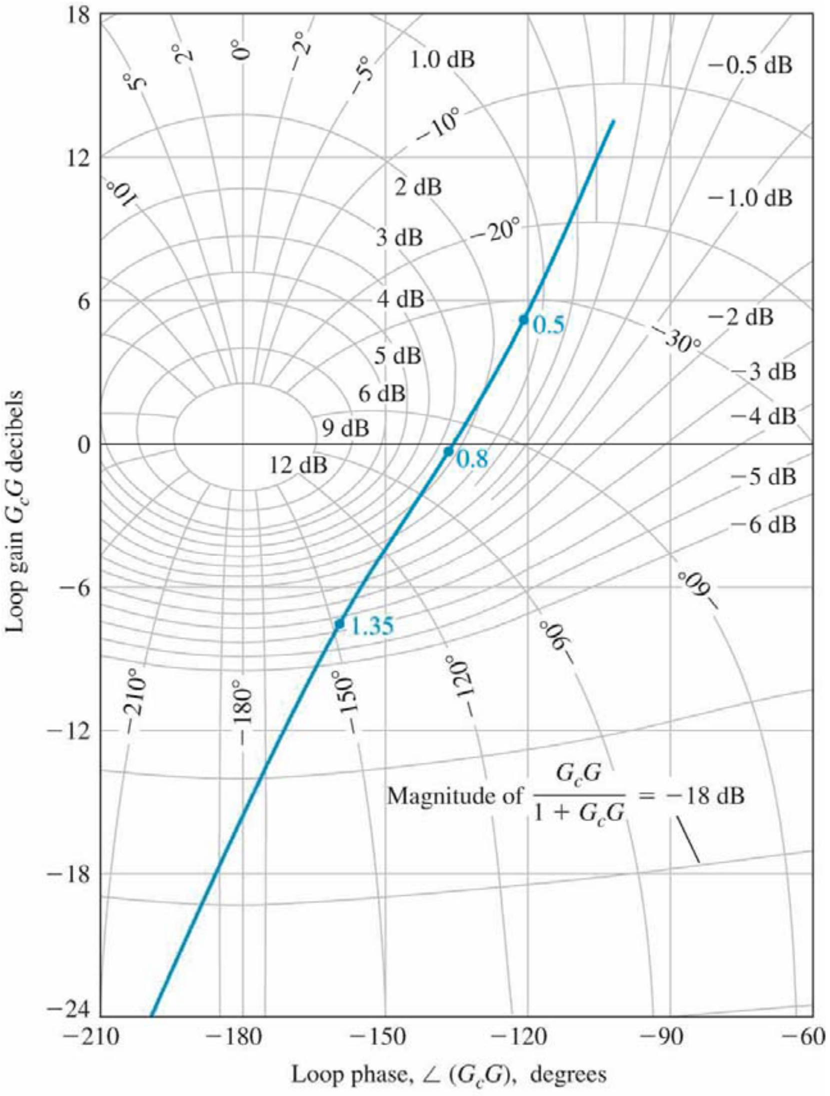



Figure 9.26 Nichols chart. The phase curves for the closed-loop system are shown as heavy curves.

Example 9.7




Resonant peak: 2.5 dB

Resonant frequency $\omega_r$: 0.8

Closed-loop phase angle at $\omega_r$: -72°

3-dB closed-loop bandwidth $\omega_B$: 1.33

Closed-loop phase angle at $\omega_B$: -142°

**Figure 9.27** Nichols diagram for $G_c(j\omega)G(j\omega) = 1/(j\omega(j\omega + 1)\,(0.2\,j\omega + 1))$. Three points on curve are shown for $\omega$ = 0.5, 0.8, and 1.35, respectively.

# 9.6 System Bandwidth

- Bandwidth of the closed-loop control system

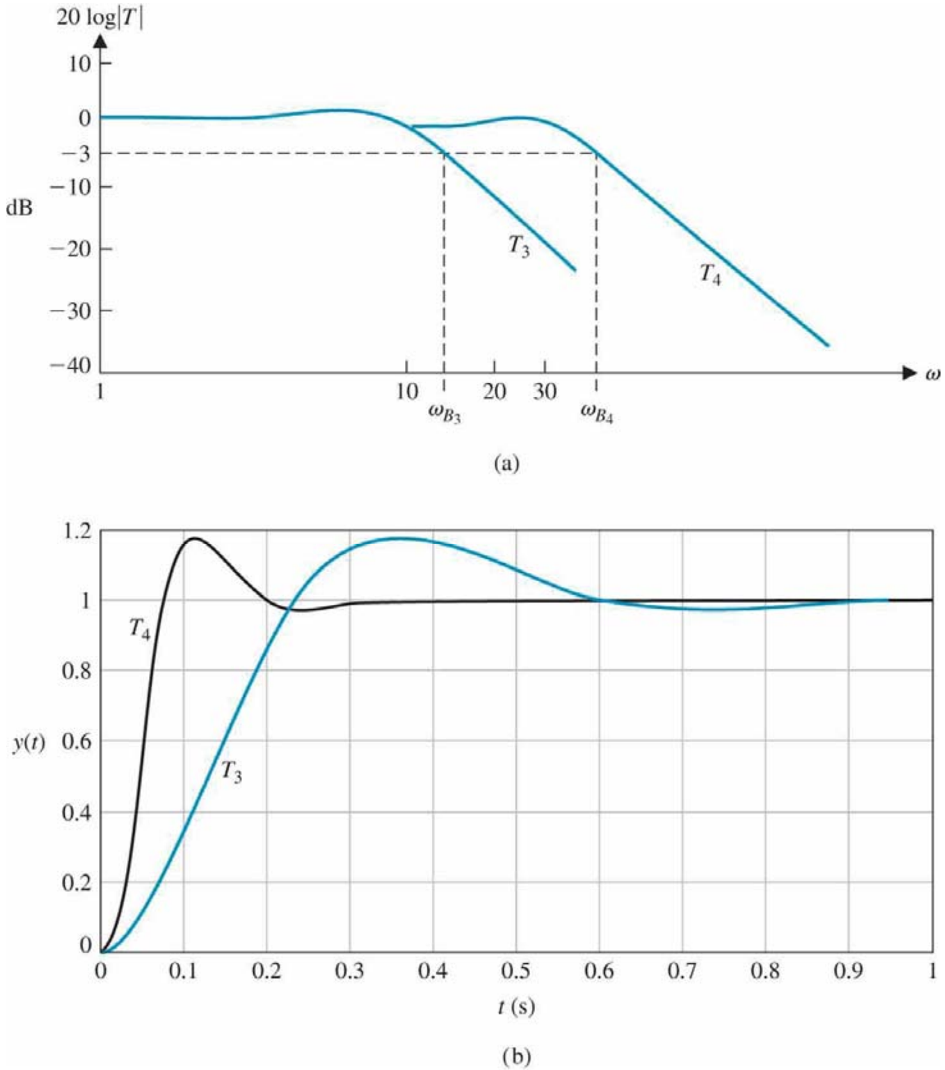
→excellent measurement of the range of fidelity (保真度) of system response (why?)

Think of this: Magnitude response of output=magnitude response of closed-loop transfer function+ magnitude response of input

→ BW is generally measured at -3 dB if low-frequency magnitude=0 dB

→$\omega_B$ is roughly proportional to peak time (speed of response)

→$\omega_B$ is inversely proportional to settling time




$$T_3(s) = \frac{100}{s^2 + 10s + 100};$$

$$T_4(s) = \frac{900}{s^2 + 30s + 900}$$

P.O.=16%

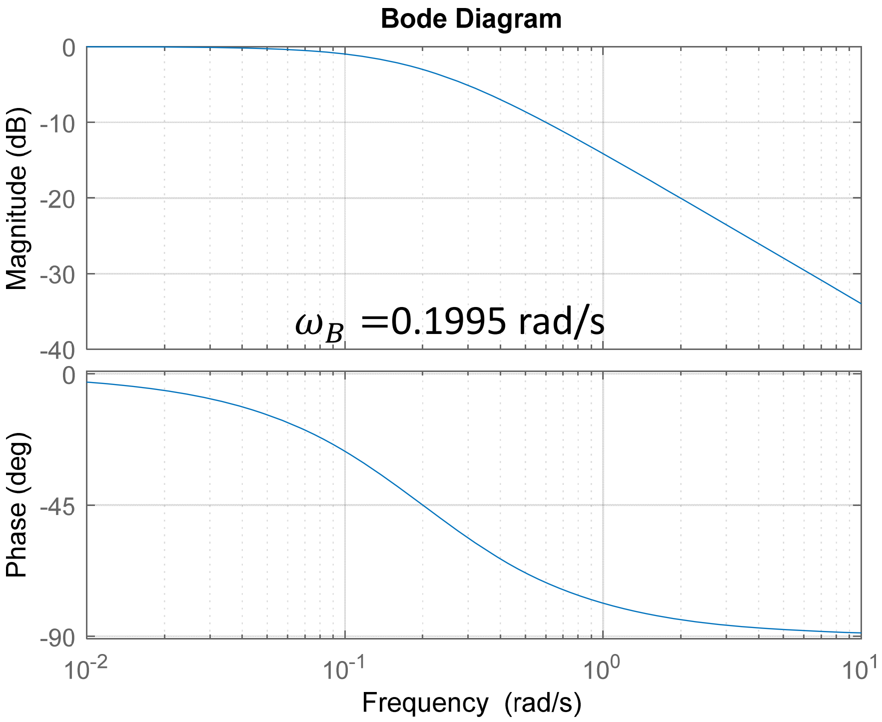
Peak time=0.12 ($T_4$), 0.36 ($T_3$)

Settling time=0.27 ($T_4$), 0.8 ($T_3$)

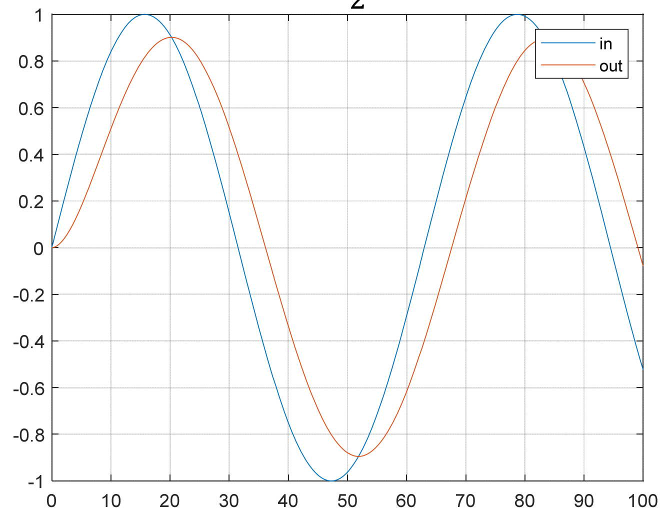
Figure Response of two second-order systems.

# Bandwidth and Fidelity

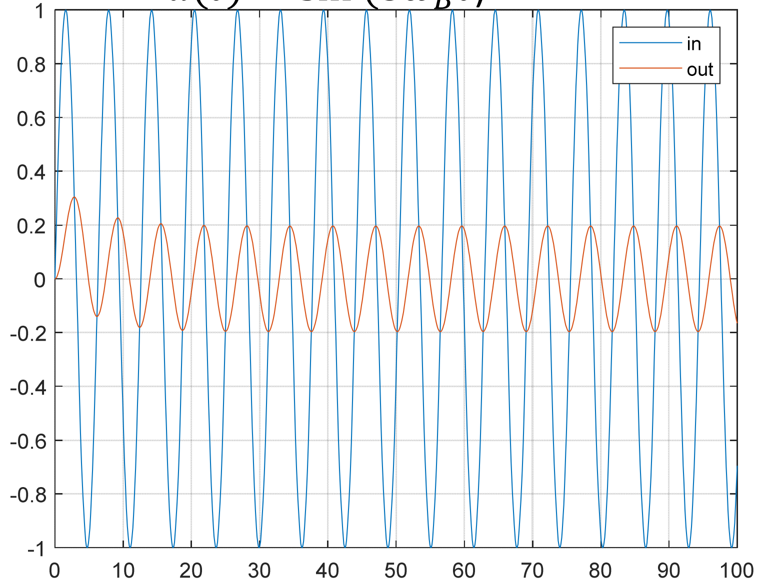
$$T(s) = \frac{1}{5s+1}$$




$$u(t) = \sin\left(\frac{\omega_B}{2} t\right)$$




$$u(t) = \sin\left(5\omega_B t\right)$$



# 9.7 The Stability of Control Systems with Time Delays

- Time delay

→ time interval between the start of an event at one point and its resulting action at another point in the system

→Nyquist criterion can be used to determine the relative stability of a system with time delay

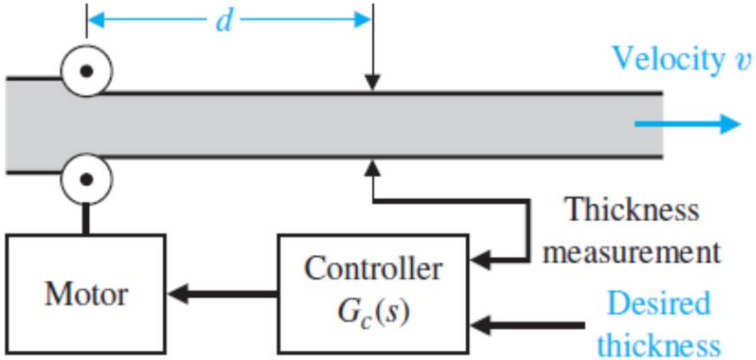
→Time delay adds a phase shift to the frequency response without altering the magnitude response

→Pade rational function approximation

- Pure time delay

→ $G_d(s) = e^{-sT}$,

- Example




$$T = \frac{d}{v}.$$

$$L(s) = G_c(s)G(s)e^{-sT}.$$

$$L(j\omega) = G_c(j\omega)G(j\omega)e^{-j\omega T}.$$

Example 9.9

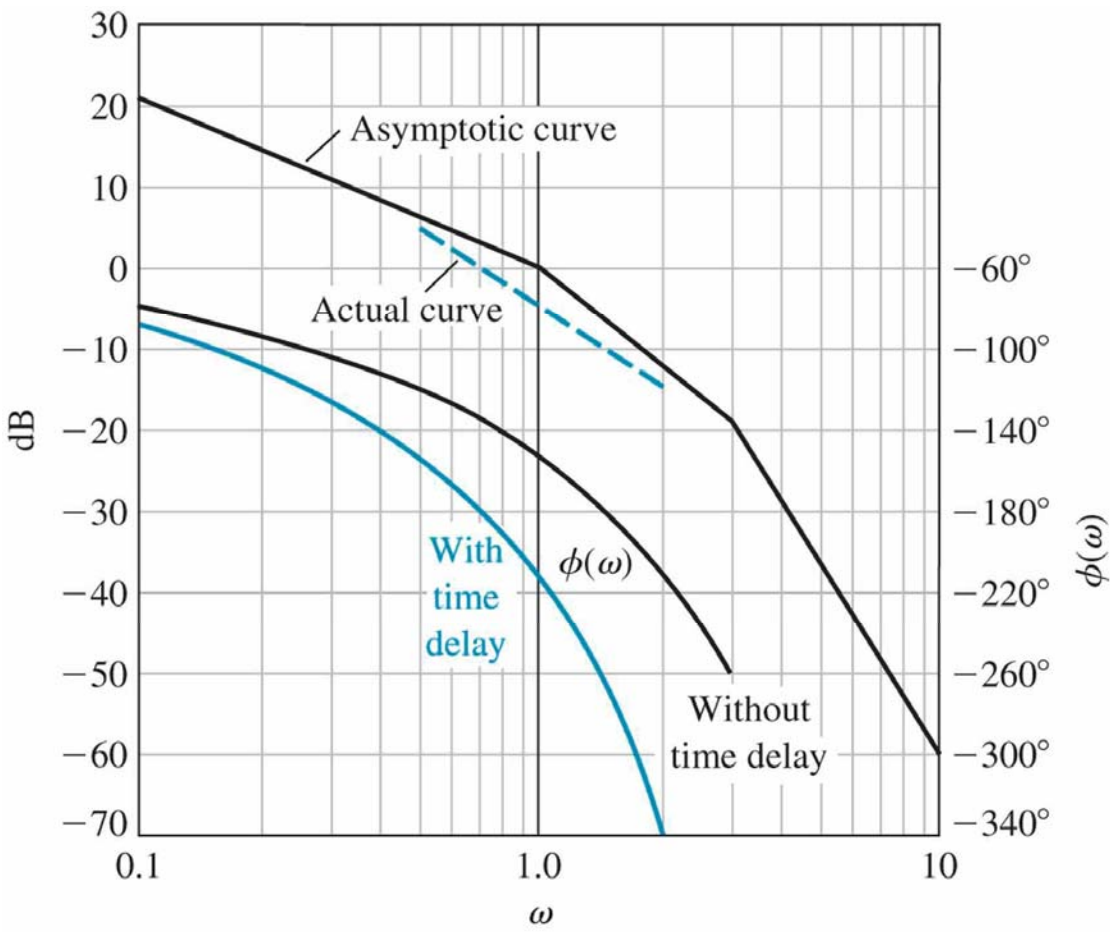
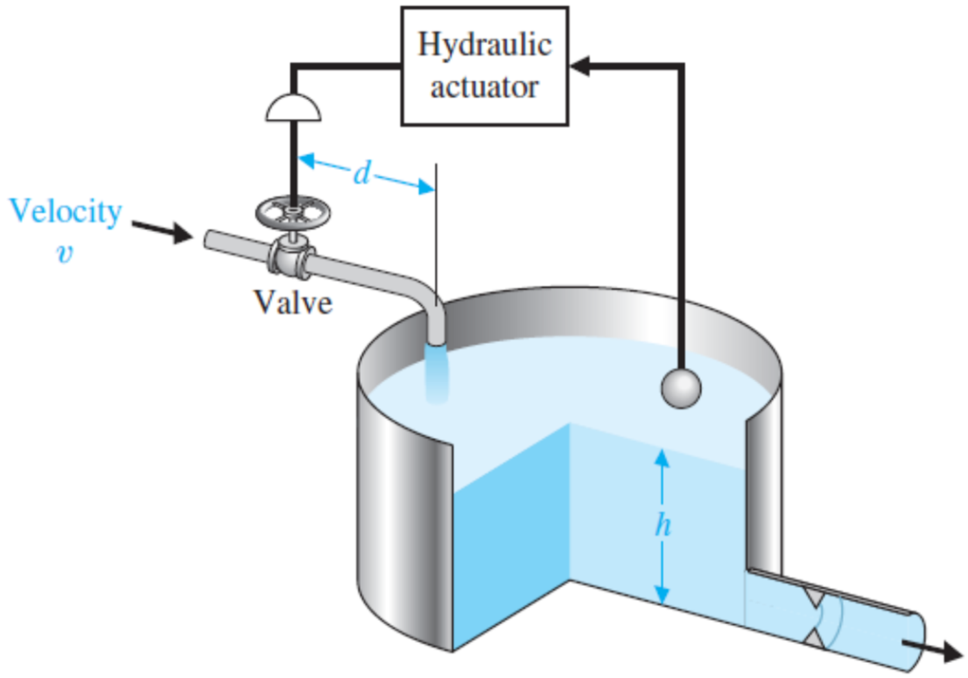
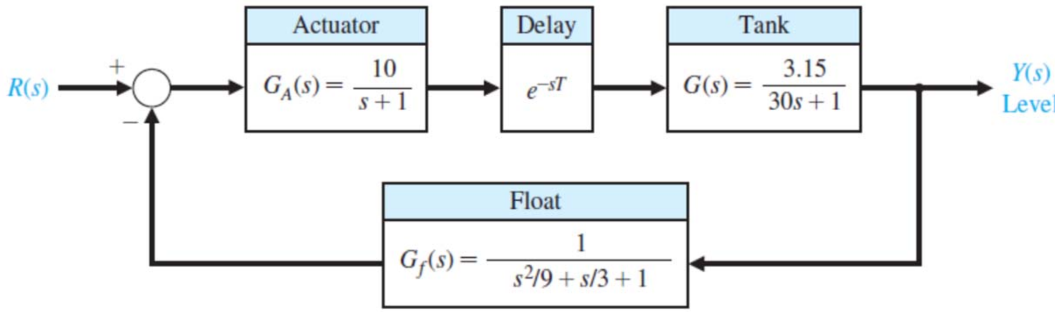



Figure 9.32 Bode diagram for level control system.



# Pade Approximation

$$e^{-sT} \approx \frac{n_1 s + n_0}{d_1 s + d_0}$$

$$e^{-sT} = 1 - sT + \frac{(sT)^2}{2!} - \frac{(sT)^3}{3!} + \frac{(sT)^4}{4!} - \frac{(sT)^5}{5!} + \cdots,$$

$$\frac{n_1 s + n_0}{d_1 s + d_0} = \frac{n_0}{d_0} + \left(\frac{d_0 n_1 - n_0 d_1}{d_0^2}\right)s + \left(\frac{d_1^2 n_0}{d_0^3} - \frac{d_1 n_1}{d_0^2}\right)s^2 + \cdots$$

$$\frac{n_0}{d_0} = 1, \frac{n_1}{d_0} - \frac{n_0 d_1}{d_0^2} = -T, \frac{d_1^2 n_0}{d_0^3} - \frac{d_1 n_1}{d_0^2} = \frac{T^2}{2}, \cdots$$

Solving for $n_0$, $d_0$, $n_1$, and $d_1$ yields

$$n_0 = d_0, d_1 = \frac{d_0 T}{2}, \text{ and } n_1 = -\frac{d_0 T}{2}.$$

Setting $d_0 = 1$ and solving yields

$$e^{-sT} \approx \frac{n_1 s + n_0}{d_1 s + d_0} = \frac{-\frac{T}{2}s + 1}{\frac{T}{2}s + 1}.$$

# 9.9 PID Controllers in Frequency Domain

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s.$$

$$G_c(s) = \frac{K_I\left(\frac{K_D}{K_I}s^2 + \frac{K_P}{K_I}s + 1\right)}{s} = \frac{K_I(\tau s + 1)\left(\frac{\tau}{\alpha}s + 1\right)}{s}.$$

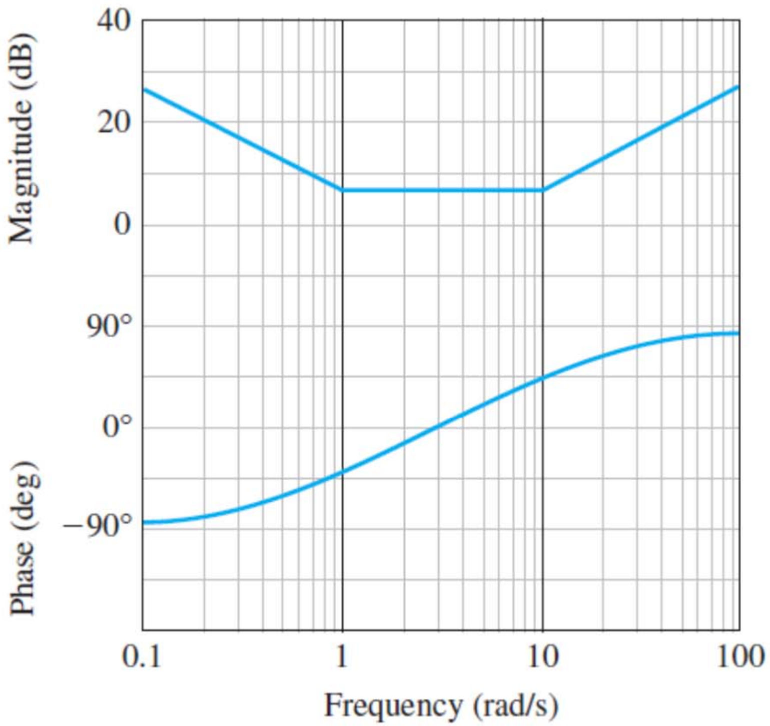



**FIGURE 9.52** Bode plot for a PID controller using the asymptomatic approximation for the magnitude curve with $K_1 = 2, \alpha = 10$, and $\tau = 1$.

PID controller is a notch (or bandstop) compensator!