Chapter 7

Root Locus Method

7.1 Introduction

Why

- Relative stability and the transient performance of a closed-loop control system

→related to the location of the closed-loop roots of the characteristic equation in the s-plane

→worthwhile to determine how the roots of the characteristic equation of a given system migrate

→useful to determine the locus of roots in the s-plane as a parameter is varied

What

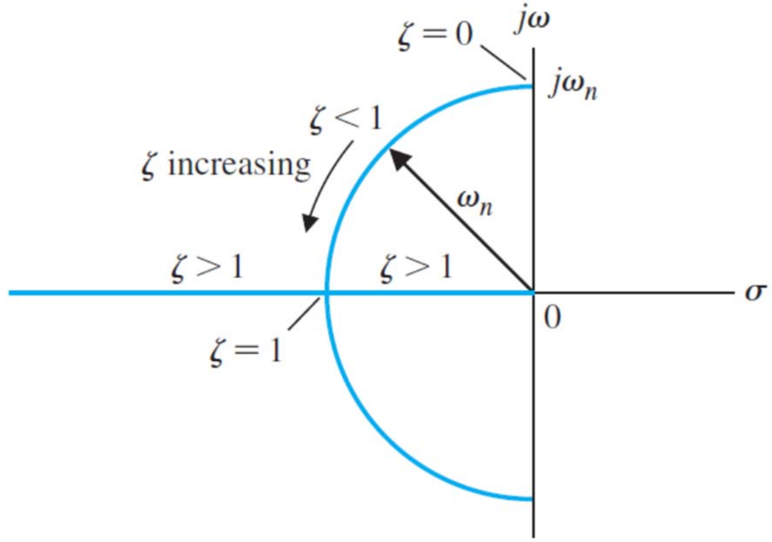
- Root locus method

→ introduced by Evans in 1948 and has been developed and utilized extensively in control engineering practice

→ a graphical method for sketching the locus of roots in the s-plane as a parameter is varied

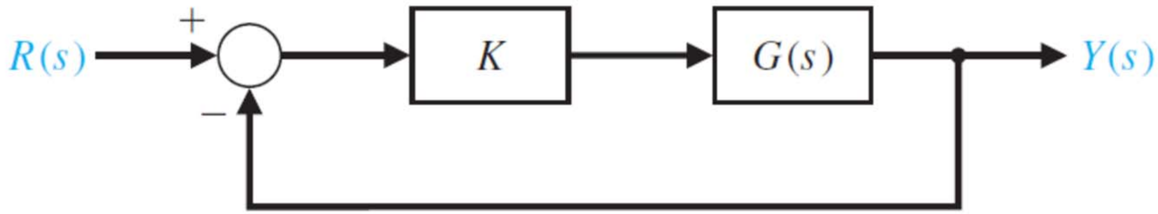
Example of Root Locus

- Discussions on damping ratio and natural frequency in Chapter 2



7.2 Root Locus Concept

$$T(s) = \frac{Y(s)}{R(s)} = \frac{p(s)}{q(s)},$$



- Characteristic equation

$$\boxed{1 + KG(s) = 0,}$$

K is a variable parameter and $0 \leq K < \infty$

$$1 + KG(s) = 0,$$

$$|KG(s)|\angle KG(s) = -1 + j0,$$

$$\boxed{|KG(s)| = 1}$$

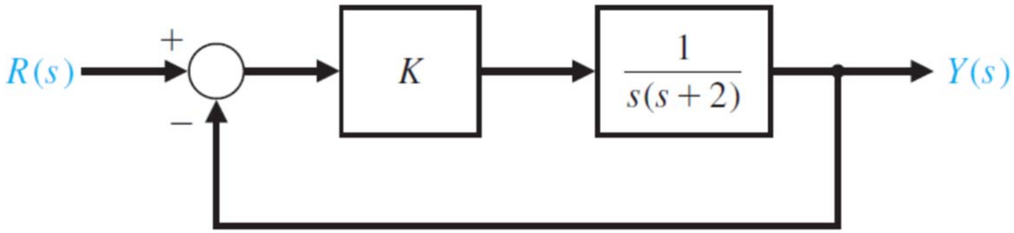
Magnitude criterion: Given s, determine K

$$\boxed{\angle KG(s) = 180° + k360°,}$$

$$k = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

Angle criterion: Determine the locus

Example

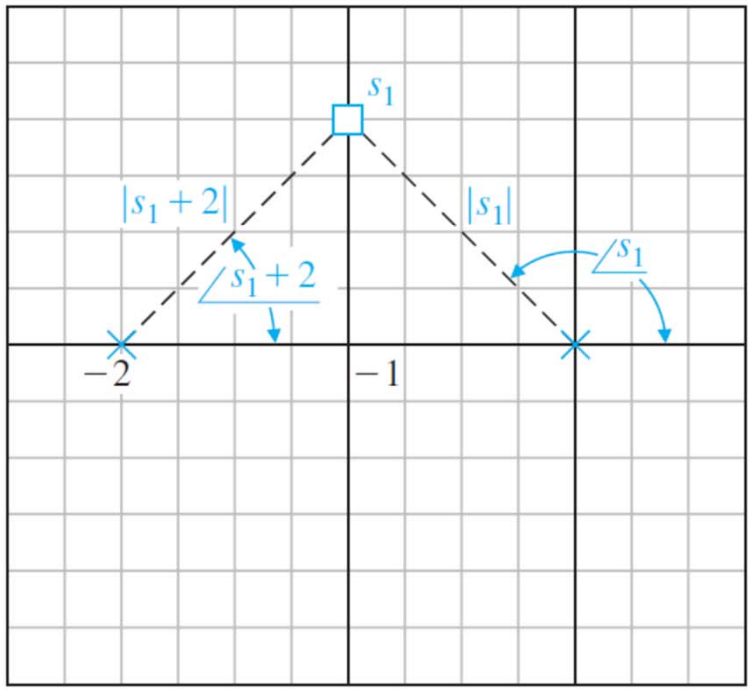


$$\Delta(s) = 1 + KG(s) = 1 + \frac{K}{s(s+2)} = 0,$$

$$|KG(s)| = \left|\frac{K}{s(s+2)}\right| = 1$$

$$\angle KG(s) = \pm 180°, \pm 540°, \ldots.$$

$$\angle \frac{K}{s(s+2)}\bigg|_{s=s_1} = -\angle s_1 - \angle(s_1+2) = -[(180° - \theta) + \theta] = -180°.$$



$$q(s) = \Delta(s) = 1 + F(s).$$

- Characteristic equation

$$1 + F(s) = 0. \quad \Rightarrow \quad F(s) = -1 + j0,$$

- Typical form of F(s)

$$F(s) = \frac{K(s + z_1)(s + z_2)(s + z_3) \cdots (s + z_M)}{(s + p_1)(s + p_2)(s + p_3) \cdots (s + p_n)}.$$

$$\Rightarrow \quad |F(s)| = \frac{K|s + z_1||s + z_2| \cdots}{|s + p_1||s + p_2| \cdots} = 1$$

$$\angle F(s) = \angle s + z_1 + \angle s + z_2 + \cdots$$
$$-(\angle s + p_1 + \angle s + p_2 + \cdots) = 180^\circ + k360^\circ,$$

7.3 Root Locus Procedure

- **Step 1:** Prepare the root locus sketch

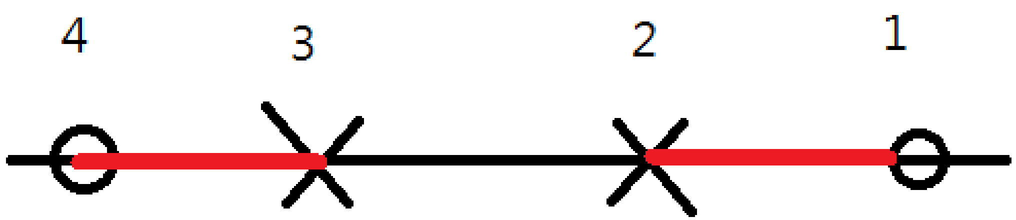
$$1 + KP(s) = 0. \qquad 0 \le K < \infty.$$

$$\Rightarrow \quad 1 + K\frac{\prod_{i=1}^{M}(s + z_i)}{\prod_{j=1}^{n}(s + p_j)} = 0.$$

Open-loop poles and zeros

1. Locus begins at the poles and ends at the zeros as *K* increases from zero to infinity
2. If # poles >= # zeros, then the number of loci = the number of poles
3. root loci must be symmetrical with respect to the horizontal real axis

- **Step 2:** Locate the segments of the real axis that are root loci. **The root locus on the real axis always lies in a section of the real axis to the left of an odd number of poles and zeros**. (angle criterion)



Example 7.1

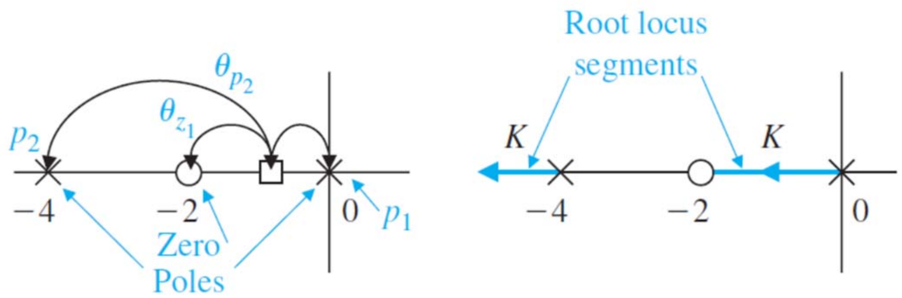
A feedback control system possesses the characteristic equation

$$1 + G_c(s)G(s) = 1 + \frac{K\left(\frac{1}{2}s + 1\right)}{\frac{1}{4}s^2 + s} = 0.$$

Step 1:

$$1 + K\frac{2(s + 2)}{s(s + 4)} = 0.$$

Step 2:



(a) (b)

- **Step 3:** The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A

centroid

$$\sigma_A = \frac{\sum \text{poles of } P(s) - \sum \text{zeros of } P(s)}{n - M}$$

360 is divided by (N-M)

angle of the asymptotes

$$\boxed{\phi_A = \frac{2k + 1}{n - M} 180^\circ,} \qquad k = 0, 1, 2, \dots, (n - M - 1)$$

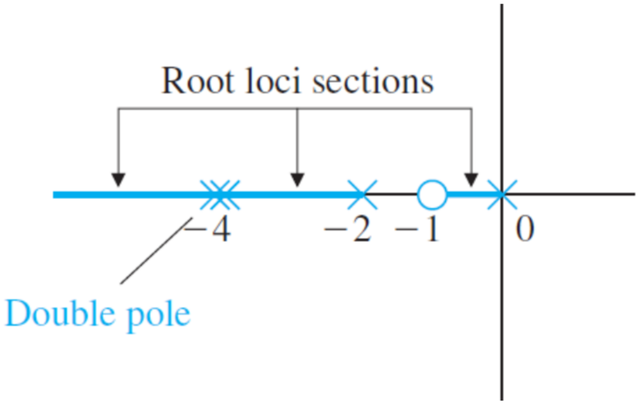
The first angle

$$\phi = \frac{180^\circ}{n - M}.$$

derived by considering a point on a root locus segment at a remote distance from the finite poles and zeros in the s-plane.

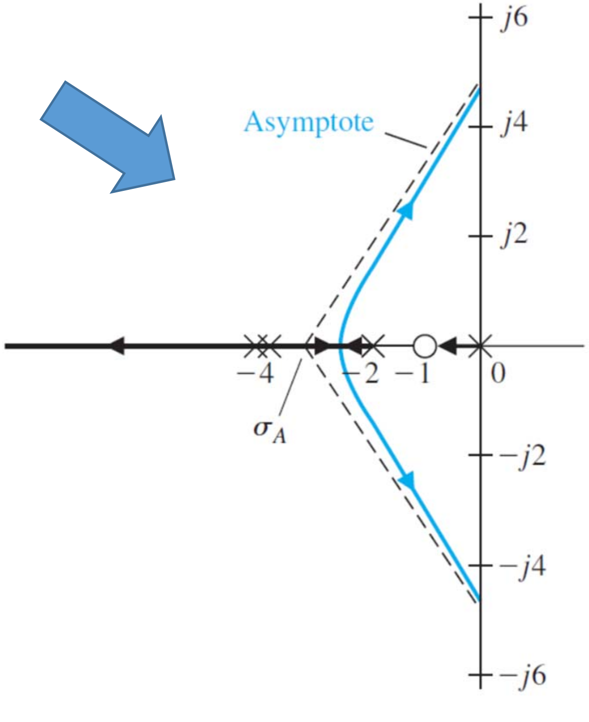
Example 7.2

$$1 + G_c(s)G(s) = 1 + \frac{K(s+1)}{s(s+2)(s+4)^2}.$$



$$\sigma_A = \frac{(-2) + 2(-4) - (-1)}{4-1} = \frac{-9}{3} = -3.$$

$$\phi_A = +60°$$



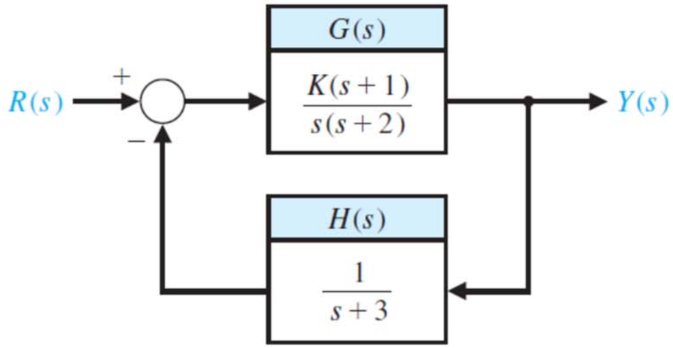
- **Step 4:** Determine where the locus crosses the imaginary axis (if it does so), using the Routh–Hurwitz criterion
- **Step 5:** Determine the breakaway point on the real axis (if any).

$$1 + KG(s) = 0,$$

$$p(s) = K.$$

breakaway point satisfies $\frac{dK}{ds} = \frac{dp(s)}{ds} = 0$

Example 7.3



$$1 + G(s)H(s) = 1 + \frac{K(s + 1)}{s(s + 2)(s + 3)} = 0.$$

$$s(s + 2)(s + 3) + K(s + 1) = 0,$$

$$p(s) = \frac{-s(s + 2)(s + 3)}{s + 1} = K.$$

$$\frac{d}{ds}\left(\frac{-s(s + 2)(s + 3)}{(s + 1)}\right) = \frac{(s^3 + 5s^2 + 6s) - (s + 1)(3s^2 + 10s + 6)}{(s + 1)^2} = 0$$

$$2s^3 + 8s^2 + 10s + 6 = 0. \qquad (7.47)$$

Table 7.1

p(s)	0	0.411	0.419	0.417	+0.390	0
s	−2.00	−2.40	−2.46	−2.50	−2.60	−3.0

- **Step 6:** Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero, using the phase angle criterion.
- **Step 7:** The final step in the root locus sketching procedure is to complete the sketch.

7 steps for sketching a root locus

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	$1 + KP(s) = 0.$
(b) Factor $P(s)$ in terms of n poles and M zeros.	$1 + K\dfrac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0.$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	$\times$ = poles, $\bigcirc$ = zeros Locus begins at a pole and ends at a zero.
(d) Determine the number of separate loci, SL.	$SL = n$ when $n \geq M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an odd number of poles and zeros.
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \dfrac{\sum(-p_j) - \sum(-z_i)}{n - M}.$ $\phi_A = \dfrac{2k+1}{n-M}180°, k = 0, 1, 2, \ldots, (n - M - 1).$
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh–Hurwitz criterion.
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$. b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex poles and the angle of locus arrival at complex zeros, using the phase criterion.	$\angle P(s) = 180° + k360°$ at $s = -p_j$ or $-z_i$.
7. Complete the root locus sketch.	

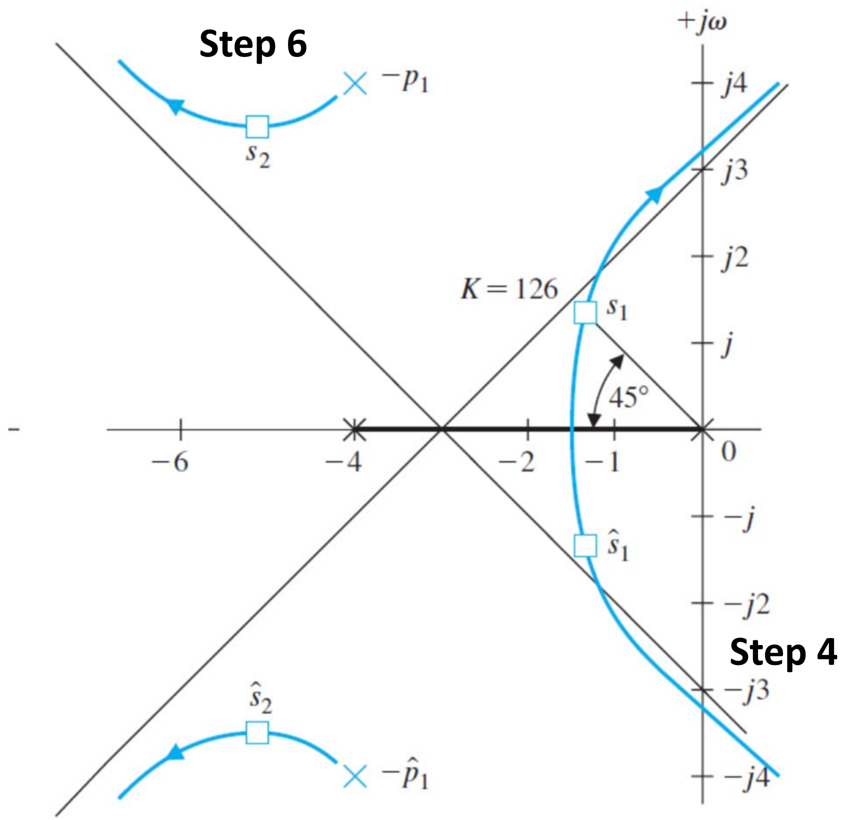
Example 7.4

$$1 + \frac{K}{s^4 + 12s^3 + 64s^2 + 128s} = 0.$$

$$1 + \frac{K}{s(s + 4)(s + 4 + j4)(s + 4 - j4)} = 0.$$

$$\phi_A = +45°, 135°, 225°, 315°.$$

$$\sigma_A = \frac{-4 - 4 - 4j - 4 + 4j}{4} = -3.$$



7.4 Sensitivity and Root Locus

- Negative feedback control

→reduce the effect of parameter variations measured by the sensitivity

→Logarithmic sensitivity (suggested by Bode)

$$S_K^T = \frac{\partial \ln T}{\partial \ln K} = \frac{\partial T/T}{\partial K/K},$$

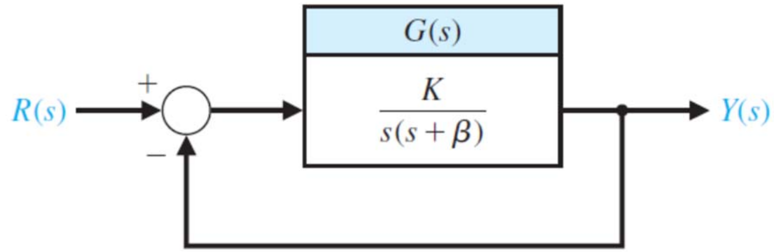
- Root sensitivity

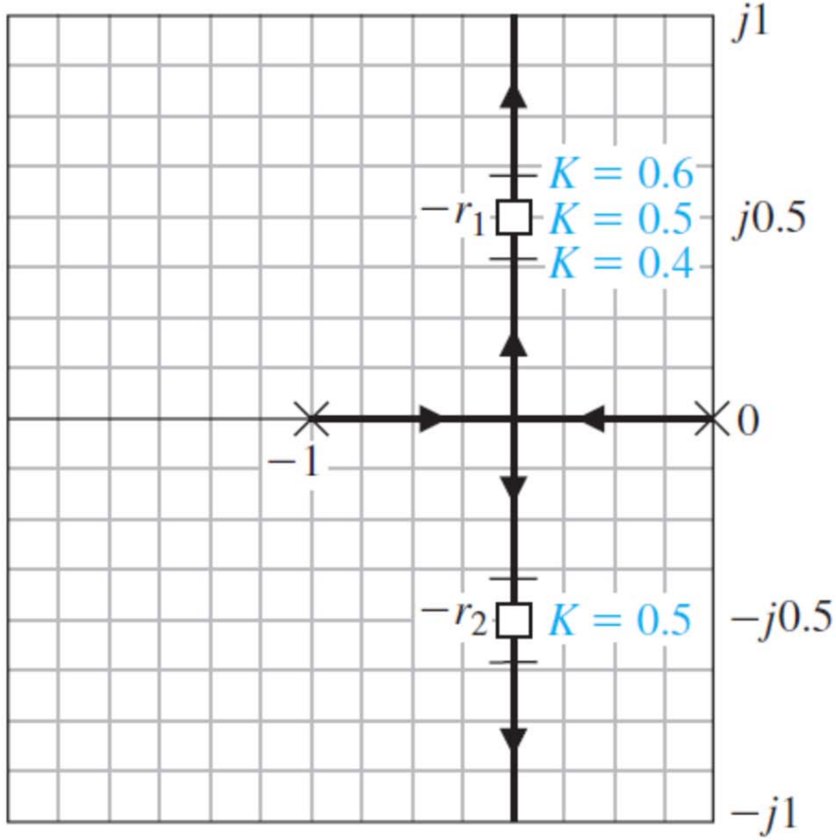
$$\boxed{S_K^{r_i} = \frac{\partial r_i}{\partial \ln K} = \frac{\partial r_i}{\partial K/K},}$$

→ K is a parameter subject to possible variations (not a controller gain)

Example 7.6

$$1 + \frac{K}{s(s + \beta)} = 0,$$





- Weakness

→The root locations represent the performance quite adequately for many systems, but due consideration must be given to the location of the zeros of the closed-loop transfer function and the dominant roots.

7.6 PID Controllers

- PID controller

$$G_c(s) = K_p + \frac{K_I}{s} + K_D s.$$

- Output equation of the controller in time domain

$$u(t) = K_p e(t) + K_I \int e(t)dt + K_D \frac{de(t)}{dt}.$$

→ Involves a proportional term, an integral term, and a derivative term

- Actual derivative term

$$G_d(s) = \frac{K_D s}{\tau_d s + 1},$$

- Proportional plus integral (PI) controller

$$\boxed{G_c(s) = K_p + \frac{K_I}{s}.}$$

- Proportional plus derivative (PD) controller

$$\boxed{G_c(s) = K_p + K_D s,}$$

PID=PI + PD

$$G_{PI}(s) = \hat{K}_P + \frac{\hat{K}_I}{s} \qquad G_{PD}(s) = \overline{K}_P + \overline{K}_D s,$$

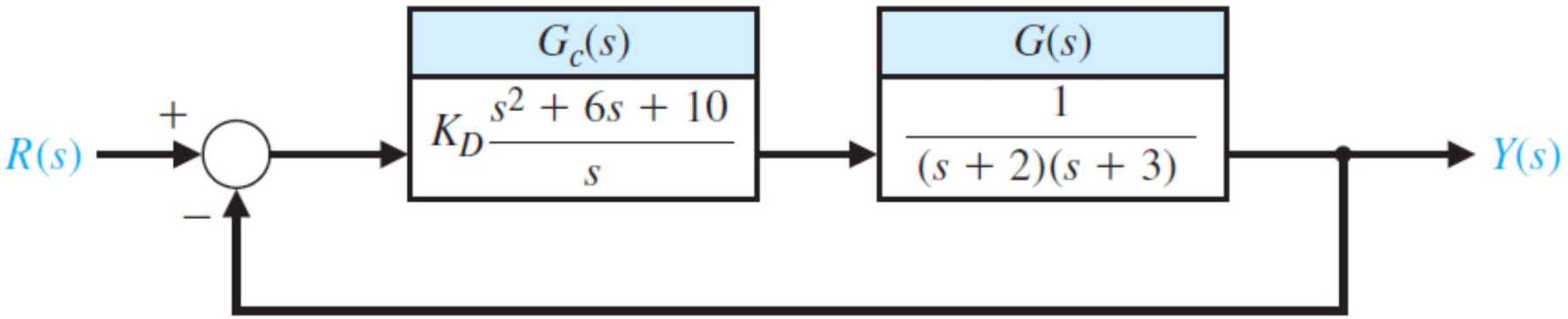
$$\begin{aligned}
G_c(s) &= G_{PI}(s)G_{PD}(s) \\
&= \left(\hat{K}_P + \frac{\hat{K}_I}{s}\right)(\overline{K}_P + \overline{K}_D s) \\
&= (\overline{K}_P\hat{K}_P + \hat{K}_I\overline{K}_D) + \hat{K}_P\overline{K}_D s + \frac{\hat{K}_I\overline{K}_D}{s} \\
&= K_P + K_D s + \frac{K_I}{s},
\end{aligned}$$

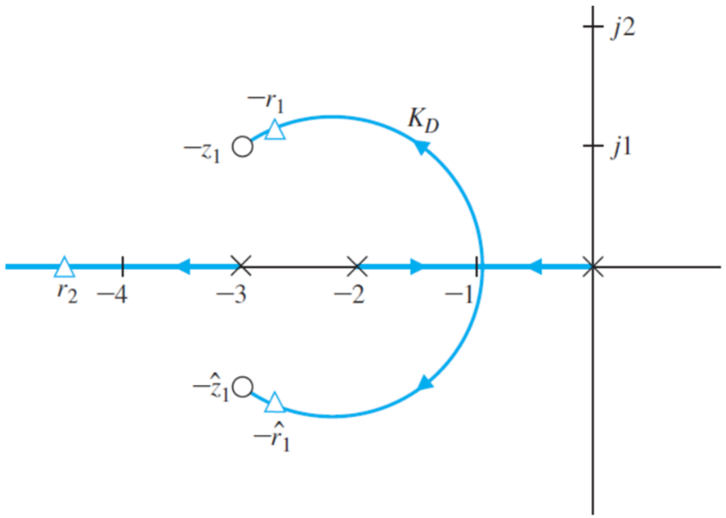
PID: one pole at the origin and 2 arbitrary zeros

$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s}$$

$$= \frac{K_D(s^2 + as + b)}{s} = \frac{K_D(s + z_1)(s + z_2)}{s},$$

Example of using PID





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PID controller can
→ Change the P.O.
→ Improve the steady-state error
→ Reduce the settling time

How?

PID tuning

- PID tuning

→The process of determining the PID gains

- Common approach

→manual PID tuning methods

→trial-and-error with minimal analytic analysis using step responses obtained via simulation

- Analytic method

→ the Ziegler–Nichols tuning method

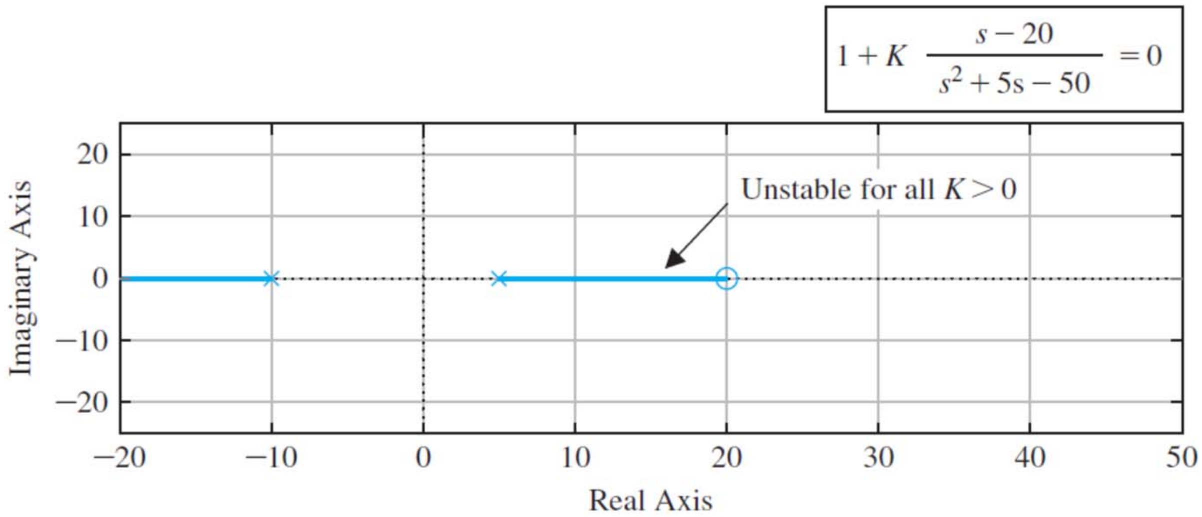
7.7 Negative Gain Root Locus

- Negative gain root locus

→ $-\infty < K \leq 0.$

Example:

$$L(s) = KG(s) = K\frac{s - 20}{s^2 + 5s - 50}$$



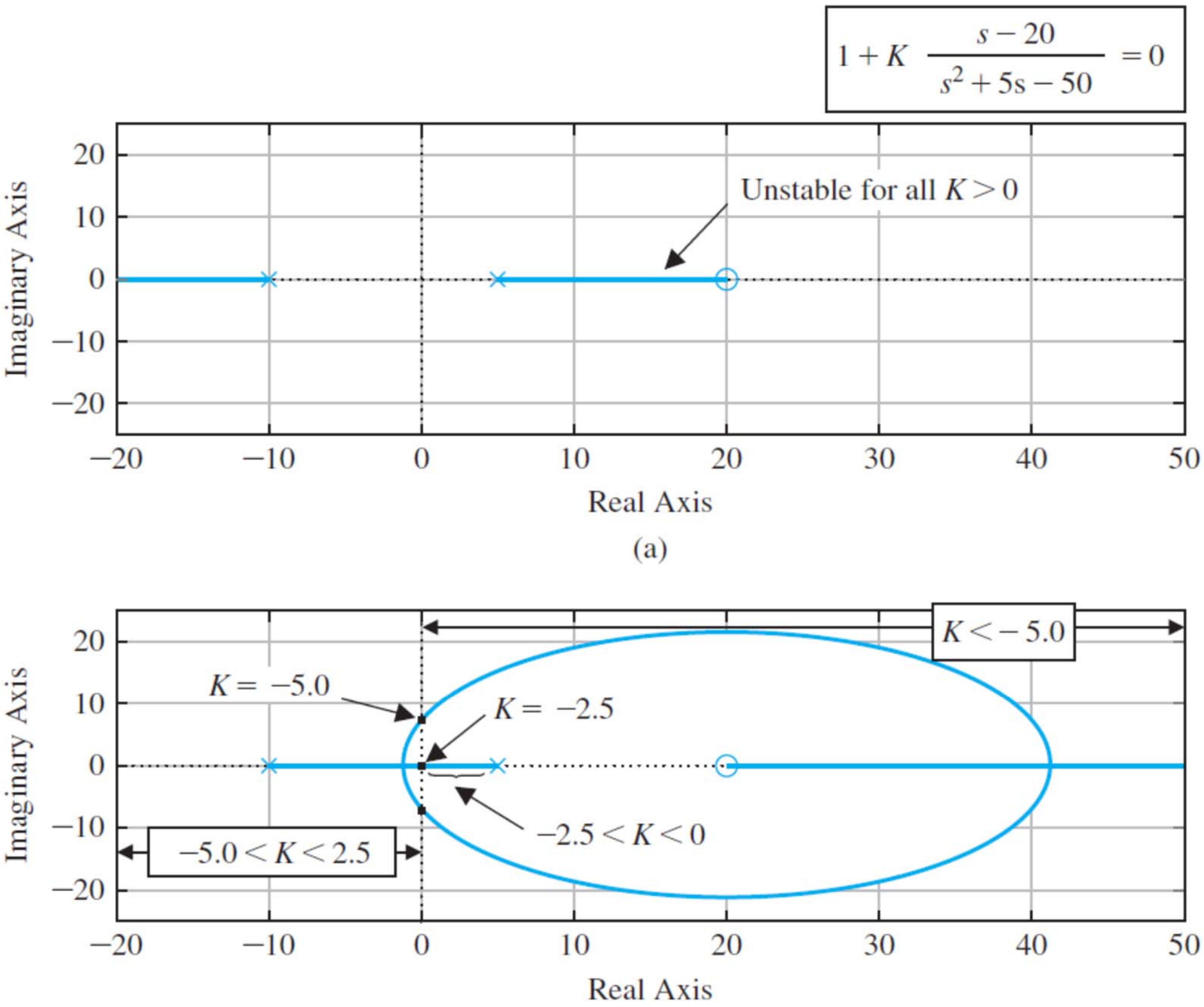
Changes for Negative Gain Root Locus

1. (segments on the real axis) Locus lies to the right of an odd number of critical frequencies of the open loop
2. (angle of the asymptotes)

$$\boxed{\phi_A = \frac{2k + 1}{n - M} 360^\circ} \qquad k = 0, 1, 2, \ldots, (n - M - 1),$$

3. (angle of departure)

$$\angle P(s) = \pm k360^\circ \text{ at } s = -p_j \text{ or } -z_i$$



$1 + K \frac{s - 20}{s^2 + 5s - 50} = 0$
Unstable for all $K > 0$
Imaginary Axis
20
10
0
−10
−20
−20
−10
0
10
20
30
40
50
Real Axis

(a)

$K < -5.0$
$K = -5.0$
$K = -2.5$
$-2.5 < K < 0$
$-5.0 < K < 2.5$
Imaginary Axis
20
10
0
−10
−20
−20
−10
0
10
20
30
40
50
Real Axis

(b)