

Chapter 6

Stability of Linear Feedback Systems

6.1 Concept of Stability

- Closed-loop feedback system that is unstable is of minimal value (exceptions: aircraft)
- Closed-loop feedback is used to
 - Stabilize an unstable systems or adjust performance of a stable open-loop system
- Absolute stability
 - Stable/not stable
- Relative stability (given a stable closed-loop system)
 - Characterize the degree of stability

Relative Stability for Aircraft Design

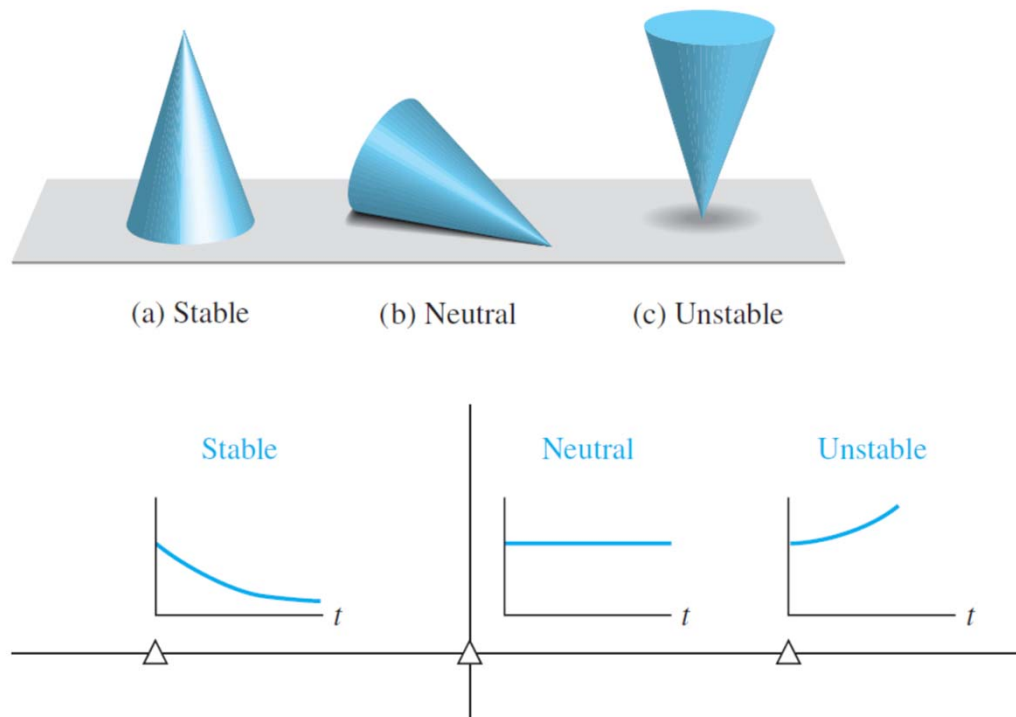
- The more stable an aircraft was, the more difficult it was to maneuver (that is, to turn)
- A acrobatic aircraft is less stable than a commercial transport; hence it can maneuver

Stability Criterion

- A system is stable (in the absolute sense) if a bounded input yields a bounded response
 - all transfer function poles lie in the left-half s-plane
 - or all the eigenvalues of the system matrix A in state variable representation lie in the left-half s-plane.
- Given that all the poles (or eigenvalues) are in the left-half s-plane
 - Examine the relative locations of the poles (or eigenvalues) for relative stability

Illustration of Stability

- The concept of stability can be illustrated by considering a right circular cone placed on a plane horizontal surface.



Stability in terms of Location of Poles

- Closed-loop transfer function

$$T(s) = \frac{p(s)}{q(s)} = \frac{K \prod_{i=1}^M (s + z_i)}{s^N \prod_{k=1}^Q (s + \sigma_k) \prod_{m=1}^R [s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)]},$$

- Output response for an impulse function input (when $N = 0$) is

$$y(t) = \sum_{k=1}^Q A_k e^{-\sigma_k t} + \sum_{m=1}^R B_m \left(\frac{1}{\omega_m} \right) e^{-\alpha_m t} \sin(\omega_m t + \theta_m),$$

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- To obtain a bounded response

→ poles of the closed-loop system must be in the left-hand portion of the s-plane.

$$T(s) = \frac{p(s)}{q(s)} = \frac{K \prod_{i=1}^M (s + z_i)}{s^N \prod_{k=1}^Q (s + \sigma_k) \prod_{m=1}^R [s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)]},$$

Stability in terms of Location of Poles (Summary)

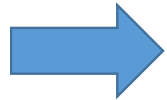
- Stable system
 - all the poles of the transfer function are in the left-half s -plane
 - bounded inputs yield bounded outputs
- Marginally stable system
 - simple poles on the imaginary axis and all other poles in the left-half s -plane
 - only certain bounded inputs (sinusoids of the frequency of the poles) will cause the output to become unbounded; other bounded inputs lead to oscillatory outputs
- Unstable system
 - at least one pole in the right-half s -plane or repeated poles on the imaginary axis
 - the output is unbounded for any input.

6.2 Routh–Hurwitz Stability Criterion

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0.$$

• Factorization $a_n (s - r_1)(s - r_2) \cdots (s - r_n) = 0,$

$$\begin{aligned} q(s) = & a_n s^n - a_n (r_1 + r_2 + \dots + r_n) s^{n-1} \\ & + a_n (r_1 r_2 + r_2 r_3 + r_1 r_3 + \dots) s^{n-2} \\ & - a_n (r_1 r_2 r_3 + r_1 r_2 r_4 \dots) s^{n-3} + \dots \\ & + a_n (-1)^n r_1 r_2 r_3 \cdots r_n = 0. \end{aligned}$$



Necessary conditions: all the coefficients are nonzero and have the same sign

- If the necessary condition is satisfied, we still need to proceed further to ascertain the stability of the system
- Example: $q(s) = (s + 2)(s^2 - s + 4) = (s^3 + s^2 + 2s + 8)$
- Routh–Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

Routh–Hurwitz Criterion

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

• Routh array

s^n	a_n	a_{n-2}	a_{n-4}	\cdots
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	\cdots
s^{n-2}	b_{n-1}	b_{n-3}	b_{n-5}	\cdots
s^{n-3}	c_{n-1}	c_{n-3}	c_{n-5}	\cdots
\vdots	\vdots	\vdots	\vdots	\vdots
s^0	h_{n-1}			

• Butterfly notations:

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \dots$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \dots$$

Four Distinct Cases for Routh Array Calculation Procedure


- Routh–Hurwitz criterion states that the number of roots with positive real parts is equal to the number of changes in sign of the first column of the Routh array.
- Four distinct cases or configurations of the first column array must be considered
 - each case must be treated separately and requires suitable modifications of the array calculation procedure
- Four cases: (1) No element in the first column is zero; (2) there is a zero in the first column, but some other elements of the row containing the zero in the first column are nonzero; (3) there is a zero in the first column, and the other elements of the row containing the zero are also zero; and (4) as in the third case, but with repeated roots on the imaginary axis

Example 6.4

- Consider the characteristic equation

$$q(s) = s^5 + s^4 + 4s^3 + 24s^2 + 3s + 63.$$

- Routh array

s^5		1	4	3	
s^4		1	24	63	
s^3		-20	-60	0.	
s^2		21	63	0	
s^1		0	0	0	

unstable

Videos for Quick Tips

- <https://www.youtube.com/watch?v=WBCZBOB3LCA&t=9s>
- <https://www.youtube.com/watch?v=oMmUPvn6IP8>
- Also please read the textbook for the detailed discussions.

6.3 Relative Stability of Feedback Control Systems

- Routh–Hurwitz criterion ascertains the absolute stability of a system by determining whether any of the roots of the characteristic equation lie in the right-half s -plane
- It is desirable to determine the relative stability
 - 1) measured by the relative real part of each root or pair of roots
 - 2) Damping ratio
- Axis shift
 - Determine the relative stability by shifting the axis

Example 6.6

$$q(s) = s^3 + 4s^2 + 6s + 4.$$

➔ $(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n^3 + s_n^2 + s_n + 1$

➔
$$\begin{array}{l|ll} s_n^3 & 1 & 1 \\ s_n^2 & 1 & 1 \\ s_n^1 & 0 & 0 \\ s_n^0 & 1 & 0 \end{array}$$

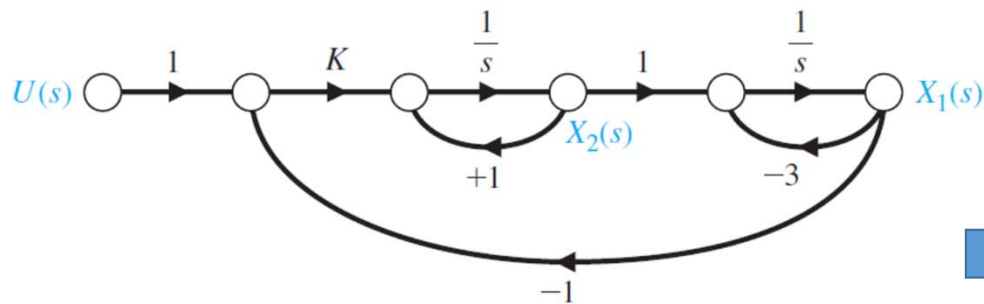
After shifting the roots to the right by one unit, we have an unstable system.

https://www.youtube.com/watch?v=wGC5C_7Yy-s&t=1s

6.4 Stability of State Variable Systems

- System represented by a signal-flow graph
 - Obtain the characteristic equation by evaluating the graph determinant (Mason's signal gain formula)
- System represented by a block diagram model
 - Obtain the characteristic equation using the block diagram reduction methods
- System represented by a state-space model
 - Obtain the characteristic equation by evaluating the eigenvalues of the transition matrix A (why?)

Example 6.7 (Mason's signal flow gain formula)



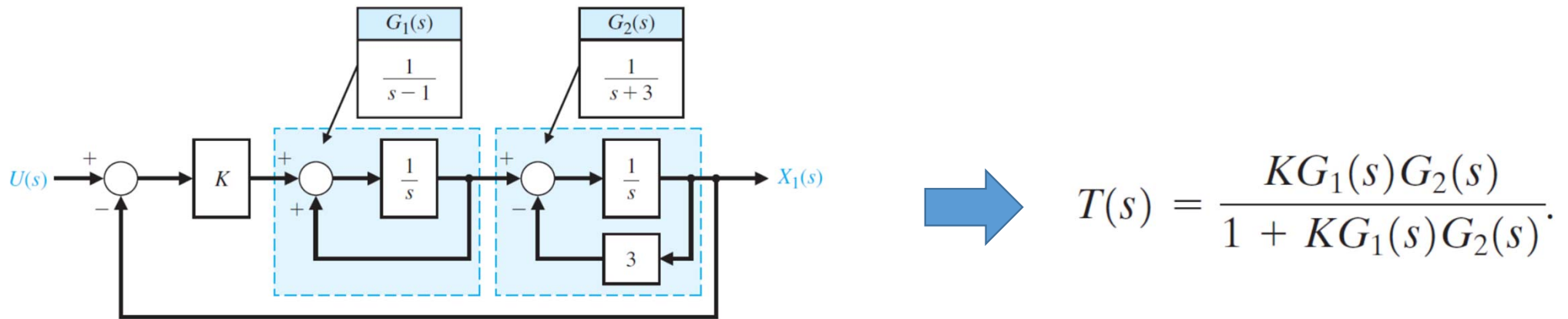
$$\dot{x}_1 = -3x_1 + x_2 \quad \text{and} \quad \dot{x}_2 = +1x_2 - Kx_1 + Ku,$$

$$\Rightarrow L_1 = s^{-1}, \quad L_2 = -3s^{-1}, \quad \text{and} \quad L_3 = -Ks^{-2},$$

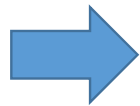
$$\Rightarrow \Delta = 1 - (L_1 + L_2 + L_3) + L_1L_2 = 1 - (s^{-1} - 3s^{-1} - Ks^{-2}) + (-3s^{-2})$$

$$\Rightarrow s^2 + 2s + (K - 3) = 0. \quad \Rightarrow \text{we require } K > 3 \text{ for stability.}$$

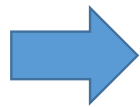
Example 6.7 (Block diagram reduction)



$$T(s) = \frac{KG_1(s)G_2(s)}{1 + KG_1(s)G_2(s)}$$



$$\Delta(s) = 1 + KG_1(s)G_2(s) = 0,$$



$$\Delta(s) = (s - 1)(s + 3) + K = s^2 + 2s + (K - 3) = 0.$$

Example 6.8 (Eigenvalues of transition matrix)

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

➔
$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det \left\{ \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -\alpha & -\beta & 0 \\ \beta & -\gamma & 0 \\ \alpha & \gamma & 0 \end{bmatrix} \right\}$$

$$= \det \begin{bmatrix} \lambda + \alpha & \beta & 0 \\ -\beta & \lambda + \gamma & 0 \\ -\alpha & -\gamma & \lambda \end{bmatrix}$$
$$= \lambda[\lambda^2 + (\alpha + \gamma)\lambda + (\alpha\gamma + \beta^2)] = 0.$$

➔ System is marginally stable when $\alpha + \gamma > 0$ and $\alpha\gamma + \beta^2 > 0$.