Midterm (I) 參考解答

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這是第一次期中考的參考解答。僅為參考，並非唯一標準。配分請依照助教或老師決定。

Problem 1. Determine the following series are convergent or divergent. Please explain why they are convergent or divergent.

i $\displaystyle\sum_{n=1}^{\infty} \tan^3 \frac{1}{\sqrt{n}}$

ii $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

iii $\displaystyle\sum_{n=1}^{\infty} \left(3 + \frac{2}{n}\right)^{-n}$

iv $\displaystyle\sum_{n=2}^{\infty} (-1)^n \sec \frac{1}{\sqrt{2n-1}}$

v $\displaystyle\sum_{n=1}^{\infty} \tanh \sqrt{n+1} - \tanh \sqrt{n}$

Solution:

i Use the fact that

$$\lim_{n\to\infty} \frac{\tan \frac{1}{n}}{\frac{1}{n}} = 1.$$

We have

$$\lim_{n\to\infty} \frac{\tan^3 \frac{1}{\sqrt{n}}}{\frac{1}{\sqrt{n^3}}} = 1$$

By the limit comparison test, since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$ converges, we obtain that $\displaystyle\sum_{n=1}^{\infty} \tan^3 \frac{1}{\sqrt{n}}$ converges as well.

ii Use the ratio test. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{((n+1)!)^3}{(3n+3)!} \cdot \frac{(3n)!}{(n!)^3} = \lim_{n\to\infty} \frac{(n+1)^3}{(3n+3)(3n+2)(3n+1)} = \frac{1}{27} < 1$$

Hence the series $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ is convergent.

iii Use the ratio test. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left(3+\frac{2}{n+1}\right)^{-n-1}}{\left(3+\frac{2}{n}\right)^{-n}}$$
$$= \lim_{n\to\infty}\left(\frac{(3n+2)/n}{(3n+4)/(n+1)}\right)^n \cdot \frac{1}{3+\frac{1}{n+1}}$$
$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1-\frac{2}{3n+4}\right)^n \cdot \frac{1}{3+\frac{1}{n+1}} = \frac{e^{1/3}}{3} < 1$$

Note: We can use L'hospital's rule to find that

$$\lim_{n\to\infty}\left(1-\frac{2}{3n+4}\right)^n = e^{-2/3}$$

Or use the definition of e and some change of variables.

Hence the series $\displaystyle\sum_{n=1}^{\infty}\left(3+\frac{2}{n}\right)^{-n}$ is convergent.

iv Notice that

$$\lim_{n\to\infty}\sec\frac{1}{\sqrt{2n-1}} = 1$$

(because it is a continuous function and tends to $\sec(0) = 1$.)

Meaning that $\displaystyle\lim_{n\to\infty}(-1)^n\sec\frac{1}{\sqrt{2n-1}}$ does not exists. Hence, (by test for divergence) the series $\displaystyle\sum_{n=2}^{\infty}(-1)^n\sec\frac{1}{\sqrt{2n-1}}$ diverges.

v Observe that it is a kind of telescoping series. We have

$$\sum_{n=1}^{k}\tanh\sqrt{n+1}-\tanh\sqrt{n}$$
$$= \cancel{\tanh(\sqrt{2})} - \tanh(1) + \cancel{\tanh(\sqrt{3})} - \cancel{\tanh(\sqrt{2})} + \cdots + \tanh(\sqrt{k+1}) - \cancel{\tanh(\sqrt{k})}$$
$$= \tanh(\sqrt{k+1}) - \tanh(1)$$

Since

$$\lim_{n\to\infty}\tanh(\sqrt{k+1}) = \lim_{n\to\infty}\frac{e^{\sqrt{k+1}}-e^{-\sqrt{k+1}}}{e^{\sqrt{k+1}}+e^{-\sqrt{k+1}}} = 1$$

We obtain

$$\sum_{n=1}^{\infty}\tanh\sqrt{n+1}-\tanh\sqrt{n} = 1-\tanh(1).$$

Hence the series is convergent.

Problem 2. Let $a_n \geq 0$ for all n. Suppose $\sum_{n=1}^{\infty} a_n^2$ converges. Show that $\sum_{n=1}^{\infty} \frac{a_n}{n}$ also converges.

Solution: Use Cauchy-Schwarz inequality. We have

$$\left(\sum_{n=1}^{k} \frac{a_n}{n}\right)^2 \leq \left(\sum_{n=1}^{k} a_n^2\right) \cdot \left(\sum_{n=1}^{k} \frac{1}{n^2}\right)$$

Since both of the series $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are convergent, (by assumption and the p-series) we obtain that the series $\sum_{n=1}^{k} \frac{a_n}{n}$ is convergent as well, by the comparison test.

Problem 3. Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)4^n}$

Solution: Recall that the Taylor series of $\arctan x$ at 0 is given by

$$\arctan x = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1$$

Compare with the desired series, we see that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)4^n} = 2\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{(2n+1)}$$

Since $\left|\frac{1}{2}\right| < 1$, we obtain that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)4^n} = 2\arctan\left(\frac{1}{2}\right)$$

Problem 4. Find the Taylor series of $\frac{2x-3}{x^2-2x+1}$ at -1. What is the radius of convergence?

Solution: 注意: 這題要求的是將級數在 $x = -1$ 這個點展開，並不是 $x = 0$。
Let $z = x + 1$. Observe that

$$\frac{2x-3}{x^2-2x+1} = \frac{2}{x-1} - \frac{1}{(x-1)^2} = \frac{2}{z-2} - \frac{1}{(z-2)^2}$$

We can rewrite it as

$$\frac{-1}{1-\frac{z}{2}} - \frac{1}{4} \cdot \frac{1}{(1-\frac{z}{2})^2}$$

(∗) Since $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$, we have

1. $\dfrac{-1}{1-\frac{z}{2}} = -\sum_{n=0}^{\infty}\left(\dfrac{z}{2}\right)^n$

2. $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n$. So $\dfrac{1}{4}\cdot\dfrac{1}{(1-\frac{z}{2})^2} = \dfrac{1}{4}\cdot\sum_{n=0}^{\infty}(n+1)\left(\dfrac{z}{2}\right)^n$

Combine these two results, we obtain

$$\frac{-1}{1-\frac{z}{2}} - \frac{1}{4}\cdot\frac{1}{(1-\frac{z}{2})^2} = -\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{4}\cdot\sum_{n=0}^{\infty}(n+1)\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty} 2^{-2-n}(5+n)z^n$$

Change $z = x+1$ back, we therefore obtain that the Taylor series of $\frac{2x-3}{x^2-2x+1}$ at -1 is

$$-\sum_{n=0}^{\infty} 2^{-2-n}(5+n)(x+1)^n$$

As for the radius of convergence, look at the argument (∗) above. To make this true, we require $\left|\frac{z}{2}\right| < 1$. Hence the radius of convergence is 2.

Problem 5. Let $f(x) = x\sin^2 x$.

i Find the Taylor series of $f(x)$ centered at $x = 0$.

ii Find the value of $f^{(11)}(0)$ and $f^{(102)}(0)$.

Solution: 這題的配分稍微有點重，原則上第一小題只要級數答錯，不管第二小題答案是否正確，最多只會給一分。(拿錯誤的結論推論得到正確的答案，這是不可能的。) 雖然配分比較重，不過大部分的同學其實都在這題得到蠻高的分數，很多人被扣分是因為不小心少了係數或負號，這樣的話會視為粗心，我只會稍微扣 2 到 3 分。

i Recall that $\cos x = \sum_{n=0}^{\infty}\dfrac{(-1)^n x^{2n}}{(2n)!}$. Since $\sin^2 x = \frac{1}{2}(1-\cos(2x))$, we have

$$\sin^2 x = \frac{x}{2} - \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}$$

So

$$x\sin^2 x = \frac{x}{2} - \frac{x}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!} = -\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}x^{2n+1}}{(2n)!}$$

ii Notice that $2 \cdot 5 + 1 = 11$ and $2 \cdot \frac{101}{2} + 1 = 102$, we have

$$\frac{f^{(11)(0)}}{11!} = -\frac{(-1)^5 2^{2\cdot 5-1}}{(2\cdot 5)!} \implies f^{(11)(0)} = 2^9 \cdot 11 = 5632$$

And since $\frac{101}{2}$ is not an integer, $f^{(102)}(0) = 0$.

Problem 6. Let $f(x) = (1+x)^{-1/3}$ defined on $(-1,1)$. Show that the remainder $R_n(x)$ converges to 0 as $n \to \infty$ for any $|x| < 1$.

這題一開始給的解法有誤，我將在後面說明錯誤的地方在哪。以下這個是正確的版本。

Solution: Let $\alpha \in \mathbb{R}$. We tend to show that for the function $f(x) = (1+x)^\alpha$, the remainder $R_n(x)$ converges to 0 as $n \to \infty$ for any $|x| < 1$. In our case, $\alpha = -1/3$.
First, observe that

$$f^{(n+1)}(t) = \alpha(\alpha-1)\cdots(\alpha-n)(1+t)^{\alpha-n-1}$$

Then we know

$$R_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt = \alpha_n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1-\alpha}}\,dt$$

where

$$\alpha_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n)}{n!}$$

Now, we claim that α_n is bounded. Notice that when $n > \frac{\alpha}{2} \implies \frac{\alpha}{n} < 2$

$$\frac{\alpha_n}{\alpha_{n-1}} = \frac{\alpha - n}{n} = \frac{\alpha}{n} - 1 < 1$$

So $\alpha_m < \alpha_{n-1}$ for all $m \geq n$ when $n > \frac{\alpha}{2}$ is large enough. This proved that α_n is bounded when n is large. Let M be one of upper bounds of α_n. This M is independent of n.
Now we discuss the convergence of $R_n(x)$ for different x.

- Suppose that $1 > x > 0$. In the formula of $R_n(x)$ we have $0 < t < x < 1$, so

$$\frac{1}{(1+t)^{n+1}} < 1 \quad \text{and} \quad (1+t)^\alpha < 2^\alpha.$$

  Therefore

$$R_n(x) = \alpha_n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1-\alpha}}\,dt \leq M \cdot 2^\alpha \int_0^x (x-t)^n\,dt$$

  Since $0 < x < 1$ we have

$$\int_0^x (x-t)^n\,dt = \left.\frac{-(x-t)^{n+1}}{n+1}\right|_0^x = \frac{x^{n+1}}{n+1} \to 0 \text{ as } n \to \infty$$

  and thus we obtain $R_n(x) \to 0$ as $n \to \infty$.

- Suppose $0 > x > -1$. Since $\frac{(x-t)^n}{(1+t)^{n+1-\alpha}}$ is continuous on $(x, 0)$, by generalized M.V.T. (for integral), there exists $r \in (x, 0)$ such that

$$\int_0^x \frac{(x-t)^n}{(1+t)^{n+1-\alpha}}\,dt = \frac{(x-r)^n}{(1+r)^{n+1-\alpha}} \int_0^x 1\,dt = \frac{x(x-r)^n}{(1+r)^{n+1-\alpha}}$$

Now, observe that

$$\begin{aligned} -1 < x < 0 &\implies -r > rx > 0 \\ -1 < x < r &\implies 1 > -x > -r \\ &\implies 1 > -x > -r > rx > 0 \end{aligned}$$

Hence we see $|-x-rx| \geq |-x+r|$, namely, we have

$$|x-r| \leq |x+rx|$$

Apply this result to our formula of $R_n(x)$, we obtain

$$\begin{aligned} |R_n(x)| &= \left|\alpha_n \int_0^x \frac{(x-t)^n}{(1+t)^{n+1-\alpha}}\,dt\right| \\ &= |M| \left|\frac{x(x-r)^n}{(1+r)^{n+1-\alpha}}\right| \\ &\leq \left|\frac{Mx^{n+1}}{(1+r)^{1-\alpha}}\right| \\ &< \left|\frac{Mx^{n+1}}{(1+x)^{1-\alpha}}\right| \to 0 \end{aligned}$$

as $n \to \infty$.

Hence $R_n(x) \to 0$ as $n \to \infty$ for ant $|x| < 1$.

這裡提供另一個方法。不過這個方法還有地方需要改進，就給有興趣的同學想看看。

Solution: Let $\alpha \in \mathbb{R}$. We tend to show that the remainder $R_n(x)$ of the function $f(x) = (1+x)^\alpha$ converges to 0 as $n \to \infty$ for any $|x| < 1$. In our case, $\alpha = -1/3$. For $x \in \mathbb{R}$ with $|x| < 1$, choose t so that $0 \leq |t| \leq |x| \leq 1$. So $|x-t| < |1+t|$ since

- for $x > 0$, one has $|x-t| = x-t < 1 < 1+t = |1+t|$.
- for $x < 0$, one has $|x-t| = t-x < t+1 = |1+t|$, since $-x < 1$.

Thus, by continuity, $\left|\frac{x-t}{1+t}\right|$ attains its maximum $q < 1$ for t on the closed interval between zero and x. Then:

$$\begin{aligned}
|R_n(x)| &= \Big|\int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)dt\Big| \\
&\le \int_0^x \Big|\frac{(x-t)^n}{n!} f^{(n+1)}(t)\Big|dt \\
&\overset{(1)}{=} \int_0^x \Big|\frac{(x-t)^n}{(1+t)^n}\cdot(\alpha-n)\cdot\binom{\alpha}{n}\cdot(1+t)^{\alpha-1}\Big|dt \\
&\overset{(2)}{\le} n\cdot\binom{\alpha}{n}\cdot|q|^n\cdot\int_0^x |\alpha/n-1|\cdot|1+t|^{\alpha-1}dt \\
&\overset{(3)}{\le} n\cdot\binom{\alpha}{n}\cdot|q|^n\cdot(|\alpha|+1)\cdot C \to 0 \text{ as } n\to\infty,
\end{aligned}$$

On the above arguments, we use the following facts:

(1) $f^{(n+1)}(t) = (n+1)!\binom{\alpha}{n+1}(1+t)^{\alpha-n-1}$ and $\binom{\alpha}{n+1} = \frac{\alpha-n}{n+1}\binom{\alpha}{n}$

(2) $\left|\frac{x-t}{1+t}\right|^n \le |q|^n$

(3) $|\alpha/n-1| \le |\alpha|+1$ as n large enough, and $C := \Big|\int_0^x |1+t|^{\alpha-1}dt\Big|$ is independent of n, namely, has no n in its formula.

還有一個問題在於: $n\cdot\binom{\alpha}{n}\cdot|q|^n$ 是不是真的會趨近 0?

Problem 7. A surface consists of all points P such that the distance from P to the plane $z=1$ is twice the distance from P to the point $(0,0,-1)$. Find an equation for this surface and identify it.

Solution: The distance L_1 from $P=(x,y,z)$ to the plane is given by $L_1 = |z-1|$. And the distance L_2 from P to the point $(0,0,-1)$ is given by

$$L_2 = \sqrt{x^2+y^2+(z+1)^2}$$

Since $L_1 = 2L_2$, we have

$$|z-1| = 2\sqrt{x^2+y^2+(z+1)^2}$$

Square both sides and then simplify it, we will obtain that

$$4x^2+4y^2+3\left(z+\frac{5}{3}\right)^2 = \frac{16}{3}$$

Hence it is an ellipsoid centered at $\left(0,0,-\frac{5}{3}\right)$.

Problem 8. Consider the curve $r(t)$ given by

$$\mathbf{r}(t) = \begin{bmatrix} \sin t^2 \\ \frac{2}{3}t^3 \\ \cos t^2 \end{bmatrix}, \quad t \in [0, 1]$$

i Find the total length of the curve.

ii Parametrize $r(t)$ by the arc-length function s.

iii Find the curvature function $\kappa(s)$ in terms of s.

iv Find the principal normal $\mathbf{N}(s)$.

Solution: 這題的計算十分複雜，需要小心每一個步驟，一個小地方錯了就前功盡棄了。

i Since $\mathbf{r}(t) = \left\langle \sin t^2, \frac{2}{3}t^3, \cos t^2 \right\rangle$, we first have

$$\mathbf{r}'(t) = \left\langle 2t\cos t^2, 2t^2, 2t\sin t^2 \right\rangle$$

So

$$|\mathbf{r}'(t)| = \sqrt{(2t\cos t^2)^2 + (2t^2)^2 + (2t\sin t^2)^2} = 2t\sqrt{t^2+1}$$

Hence the total length of the curve is

$$L = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 2t\sqrt{t^2+1}\, dt = \frac{2}{3}(\sqrt{8}-1)$$

by letting $u = t^2 + 1 \implies du = 2t\, dt$.

ii The arc-length function $s(t)$ is given by

$$s(t) = \int_0^t |\mathbf{r}'(w)|\, dw = \int_0^t 2w\sqrt{w^2+1}\, dw = \frac{2}{3}(\sqrt{(t^2+1)^3} - 1)$$

Hence we have

$$t(s) = \sqrt{\left(\frac{3}{2}s + 1\right)^{2/3} - 1}$$

And we can reparametrize this curve using s,

$$\mathbf{r}(s) = \begin{bmatrix} \sin\left(\left(\frac{3}{2}s+1\right)^{2/3} - 1\right) \\ \frac{2}{3}\left(\sqrt{\left(\frac{3}{2}s+1\right)^{2/3} - 1}\right)^3 \\ \cos\left(\left(\frac{3}{2}s+1\right)^{2/3} - 1\right) \end{bmatrix}, \quad s \in \left[0, \frac{2}{3}(\sqrt{8}-1)\right]$$

iii Since now the curve is parametrized by arc-length, we know $\kappa(s) = |\mathbf{T}'(s)| = |\mathbf{r}''(s)|$. So we only need to compute $|\mathbf{r}''(s)|$. First, we see

$$\mathbf{T}(s) = \mathbf{r}'(s) = \begin{bmatrix} \left(\frac{3}{2}s+1\right)^{-1/3}\cos\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right) \\ \left(\frac{3}{2}s+1\right)^{-1/3}\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right)^{1/2} \\ -\left(\frac{3}{2}s+1\right)^{-1/3}\sin\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right) \end{bmatrix}, \quad s \in \left[0, \frac{2}{3}(\sqrt{8}-1)\right]$$

And differentiate it again, we obtain

$$\mathbf{T}'(s) =$$

$$\left(\frac{3}{2}s+1\right)^{-2/3} \begin{bmatrix} -\frac{1}{2}\left(\frac{3}{2}s+1\right)^{-2/3}\cos\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right) - \sin\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right) \\ \frac{1}{2}\left(\frac{3}{2}s+1\right)^{-2/3}\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right)^{-1/2} \\ \frac{1}{2}\left(\frac{3}{2}s+1\right)^{-2/3}\sin\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right) - \cos\left(\left(\frac{3}{2}s+1\right)^{2/3}-1\right) \end{bmatrix}$$

Let $A = \left(\frac{3}{2}s+1\right)^{2/3}$ Then we can rewrite $\mathbf{T}'(s)$ as

$$\frac{2A-1}{2A\sqrt{A(A-1)}} \cdot \left(\frac{2\sqrt{A(A-1)}}{2A-1} \begin{bmatrix} -\frac{1}{2}A^{-1}\cos(A-1) - \sin(A-1) \\ \frac{1}{2}A^{-1}(A-1)^{-1/2} \\ \frac{1}{2}A^{-1}\sin(A-1) - \cos(A-1) \end{bmatrix} \right)$$

Check (carefully) that the vector

$$\frac{2\sqrt{A(A-1)}}{2A-1} \begin{bmatrix} -\frac{1}{2}A^{-1}\cos(A-1) - \sin(A-1) \\ \frac{1}{2}A^{-1}(A-1)^{-1/2} \\ \frac{1}{2}A^{-1}\sin(A-1) - \cos(A-1) \end{bmatrix}$$

is a unit vector (having length=1). Since we know $\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}(\mathbf{s})$, we hence obtain that

$$\kappa(s) = \frac{2A-1}{2A\sqrt{A(A-1)}}, \quad \text{where } A = \left(\frac{3}{2}s+1\right)^{2/3}$$

iv By the computation above, we also know that

$$\mathbf{N}(s) = \frac{2\sqrt{A(A-1)}}{2A-1} \begin{bmatrix} -\frac{1}{2}A^{-1}\cos(A-1) - \sin(A-1) \\ \frac{1}{2}A^{-1}(A-1)^{-1/2} \\ \frac{1}{2}A^{-1}\sin(A-1) - \cos(A-1) \end{bmatrix}, \quad \text{where } A = \left(\frac{3}{2}s+1\right)^{2/3}$$

Problem 9. Let $C(t) = (x(t), y(t), z(t))$, $t \in [a, b]$ be a differentiable curve in $\mathbb{R}^3$ that is parametrized by the arc length.

Let $\mathbf{T}(t)$ denote the unit tangent vector of C at $(x(t), y(t), z(t))$, and let $\mathbf{N}(t)$ be the principal normal vector of C at $(x(t), y(t), z(t))$. Define $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Show that $\frac{d\mathbf{B}}{dt}$ is parallel to $\mathbf{N}$.

Solution:

Method 1: (a) Note that since t is the arc length parameter, both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are a unit vector. So is $\mathbf{B}(t)$. Since $\mathbf{B}(t) \cdot \mathbf{B}(t) = 1$, differentiate both sides we obtain

$$\frac{d}{dt}(\mathbf{B}(t) \cdot \mathbf{B}(t)) = 2\left(\frac{d}{dt}\mathbf{B}(t)\right) \cdot \mathbf{B}(t) = 0$$

Hence $\frac{d\mathbf{B}}{dt}$ is perpendicular to $\mathbf{B}$.

(b) From definition we know $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Thus $\mathbf{B} \cdot \mathbf{T} = 0$. Differentiate both sides, we obtain

$$\frac{d}{dt}(\mathbf{B}(t) \cdot \mathbf{T}(t)) = \left(\frac{d}{dt}\mathbf{B}(t)\right) \cdot \mathbf{T}(t) + \mathbf{B}(t) \cdot \left(\frac{d}{dt}\mathbf{T}(t)\right) = 0$$

But $\frac{d}{dt}\mathbf{T}(t) = \mathbf{T}'(t)$ is parallel to $\mathbf{N}(t)$, which is perpendicular to $\mathbf{B}(t)$, we see

$$\mathbf{B}(t) \cdot \left(\frac{d}{dt}\mathbf{T}(t)\right) = 0$$

and hence

$$\left(\frac{d}{dt}\mathbf{B}(t)\right) \cdot \mathbf{T}(t) = 0$$

Meaning, $\frac{d\mathbf{B}}{dt}$ is perpendicular to $\mathbf{T}$.

(c) Since $\mathbf{B}, \mathbf{N}, \mathbf{T}$ are perpendicular to each other, and by (a) and (b) we know $\frac{d\mathbf{B}}{dt}$ is perpendicular to $\mathbf{B}$ and $\mathbf{T}$ already, we know that $\frac{d\mathbf{B}}{dt}$ has to be parallel to $\mathbf{N}$. (Because $\frac{d\mathbf{B}}{dt}$ is thus parallel to the direction of $\mathbf{B} \times \mathbf{T}$, which is exactly the same as the parallel direction of $\mathbf{N}$.)

Method 2: Use the Frenet-Serret formulas, which said

$$d\mathbf{T}/dt = \kappa\mathbf{N}$$
$$d\mathbf{N}/dt = -\kappa\mathbf{T} + \tau\mathbf{B}$$
$$d\mathbf{B}/dt = -\tau\mathbf{N}$$

By definition of $\mathbf{B}$, differentiate both sides we obtain

$$\frac{d\mathbf{B}}{dt} = \left(\frac{d\mathbf{T}}{dt}\right) \times \mathbf{N} + \mathbf{T} \times \left(\frac{d\mathbf{N}}{dt}\right)$$

Since $\frac{d\mathbf{T}}{dt} = \mathbf{N}$ and $\frac{d\mathbf{N}}{dt} = -\kappa\mathbf{T} + \tau\mathbf{B}$, we have

$$\frac{d\mathbf{B}}{dt} = \tau\mathbf{T} \times \mathbf{B}$$

Hence $\frac{d\mathbf{B}}{dt}$ is parallel to $\mathbf{N}$.

以下是第六題原本錯誤的證明:

Solution: It is easy to see that for $n \in \mathbb{N}$,

$$f^{(n)}(x) = \frac{(-1)^n 1 \cdot 4 \cdot 7 \cdots (1 - 3(n-1))}{3^n} (1+x)^{-(1+3n)/3}$$

Thus $f^{(n)}(x)$ is bounded on $(-1,1)$ except possibly near the point $x = -1$,

To avoid this problem, we can choose a small $\varepsilon > 0$ so that $I_\varepsilon = [-1+\varepsilon, 1-\varepsilon] \subseteq (-1,1)$. Then by extreme value theorem, since $f^{(n)}(x)$ is continuous on the interval I_ε, which is a finite closed interval, $f^{(n)}(x)$ attains its maximum M_ε somewhere in I_ε. Thus, from Taylor's inequality, we have

$$|R_n(x)| < \frac{M_\varepsilon}{(n+1)!} |x - 0|^{n+1} \to 0 \text{ as } n \to \infty$$

for all $x \in I_\varepsilon$. This is true for all small $\varepsilon > 0$. Hence we are done.

Remark 1.1. Use the language of set theory, we can then say the set S on which R_n converges to 0 as $n \to \infty$ is given by

$$S = \bigcup_{\varepsilon > 0 \text{ small}} I_\varepsilon$$

Actually, it is not hard to see $S = (-1,1)$.

Remark 1.2. 這題錯誤的地方在於，原先給的 M_ε 並不是固定的。而是會因為 ε 以及 n 變動。事實上當 ε 接近 0 的時候，這個 M_ε 會直接發散。所以這樣的估計並不是好的。另外，這個證明其實跟另一個定理的證明稍微類似，是關於 power series 在閉區間上的「均勻收斂性」類似。不過因為有些超出範圍，且實際上我們也不是這樣證明的，所以以後有機會的話再跟各位聊。造成各位的不便，真的很抱歉。