Midterm Exam 2

1 Let $f(x,y) = \dfrac{\sin(x^2y^2)}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

i (5 points) Find $\dfrac{\partial f(x,y)}{\partial x}$ and $\dfrac{\partial f(x,y)}{\partial y}$ for all $(x,y) \in \mathbb{R}^2$.

ii (5 points) Is f differentiable at $(0,0)$? Why or why not?

2 (10 points) Find a plane equation of the tangent plane to the surface $yz^2 = \ln(x+z)$ at the point $(0,0,1)$.

3 Let $v(x,y) = e^x \cos y$ defined on $\mathbb{R}^2$.

i (4 points) Find $\Delta v = v_{xx} + v_{yy}$.

ii (6 points) Find $u(x,y)$ on $\mathbb{R}^2$ such that $u(0,0) = 1$ and $u_x = -v_y$, $u_y = v_x$.

iii (5 points) Let $w(x,y) = e^x \cosh y$. Can you find $z(x,y)$ on $\mathbb{R}^2$ such that $z_x = -w_y$, $z_y = w_x$? Why or why not?

4 (5 points) Let $z = f(x,y)$ be a function with continuous second order partial derivatives. Define $x = s+t$ and $y = s-t$. Show that

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial s \partial t}$$

5 (10 points) Consider the polar coordinate $x = r\cos\theta$, $y = r\sin\theta$. Express $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$ in polar coordinate.

6 (10 points) Find $\displaystyle\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1+\cos^2 x}\, dx\, dy$.

7 (10 points) Find the surface area of the part of the surface $z = xy$ that lies within the cylinder $x^2+y^2 = 1$.

8 (10 points) Find the volume of the solid that is enclosed by cylinders $x^2+y^2 = 4$ and $x^2+z^2 = 4$.

9 (10 points) Suppose $u(x, y)$ is a function of two variables defined on the unit disk

$$D = \{(x, y) | x^2 + y^2 < 1\}.$$

Suppose u is differentiable on D and $\nabla u(x, y) = [0, 0]$ on D. Show that u is constant on D. (Hint: For (a, b) and (c, d) in D, consider the segment to link these two points and define $g(t) = u((1-t)a + tc, (1-t)b + td)$. Use the mean value theorem on $g(t)$.)

10 Let $f(x, y) = (1 - xy)(x + y)$ be a function of two variables.

- i (10 points) Find and classify all critical points of f in $\mathbb{R}^2$.
- ii (10 points) Find the extreme values of f on the disk $x^2 + y^2 \leq 1$. (Hint: You can use Lagrange multiplier to find the extreme value on the boundary. You might find the identity $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$ is useful.)