Midterm Exam 1

1 (25 points) Determine the following series are convergent or divergent. Please explain why they are convergent or divergent.

i $\displaystyle\sum_{n=1}^{\infty} \tan^3 \frac{1}{\sqrt{n}}$

ii $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$

iii $\displaystyle\sum_{n=1}^{\infty} \left(3+\frac{2}{n}\right)^{-n}$

iv $\displaystyle\sum_{n=2}^{\infty} (-1)^n \sec \frac{1}{\sqrt{2n-1}}$

v $\displaystyle\sum_{n=1}^{\infty} \left(\tanh\sqrt{n+1} - \tanh\sqrt{n}\right)$ (Hint: $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.)

2 (10 points) Let $a_n \geq 0$ for all n. Suppose $\displaystyle\sum_{n=1}^{\infty} a_n^2$ converges. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n}$ also converges. (Hint: Cauchy-Schwarz inequality).

3 (10 points) Find the value of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)4^n}$. (Hint: Consider the Taylor series of $\arctan x$)

4 (10 points) Find the Taylor series of $\dfrac{2x-3}{x^2-2x+1}$ at -1. What is the radius of convergence?

5 Let $f(x) = x\sin^2 x$.

i (5 points) Find the Taylor series of $f(x)$ centered at $x = 0$. (Hint: Consider $\cos 2x$.)

ii (5 points) Find the values of $f^{(11)}(0)$ and $f^{(102)}(0)$.

6 (10 points) Let $f(x) = (1+x)^{-1/3}$ define on $(-1, 1)$. Show that the remainder $R_n(x)$ converges to 0 as $n \to \infty$ for any $|x| < 1$.

7 (10 points) A surface consists of all points P such that the distance form P to the plane $z = 1$ is twice the distance from P from the point $(0, 0, -1)$. Find an equation for this surface and identify it. (Ellipsoid, elliptic paraboloid, hyperbolic paraboloid, hyperboloid of one sheet or hyperboloid of two sheets.)

8 Consider the curve $r(t)$

$$\mathbf{r}(t) = \begin{bmatrix} \sin t^2 \\ \frac{2}{3}t^3 \\ \cos t^2 \end{bmatrix}, \quad t \in [0, 1]$$

i (5 points) Find the total length of the curve.

ii (5 points) Parametrize $r(t)$ by the arc length function s.

iii (5 points) Find the curvature function $\kappa(s)$ in terns of s.

iv (5 points) Find the principle normal $\mathbf{N}(s)$.

9 (5 points) Let $C(t) = (x(t), y(t), z(t))$, $t \in [a, b]$ be a differentiable curve in $\mathbb{R}^3$ that is parametrized by the arc length. Let $\mathbf{T}(t)$ denote the unit tangent vector of C at $(x(t), y(t), z(t))$, and let $\mathbf{N}(t)$ be the principal normal vector of C at $(x(t), y(t), z(t))$. Define $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Show that $\dfrac{d\mathbf{B}}{dt}$ is parallel to $\mathbf{N}$.