Lesson 2
Time-varying fields

Chen-Bin Huang

Department of Electrical Engineering
Institute of Photonics Technologies
National Tsing Hua University, Taiwan

Outline

- Review: wave equations in source-free region
- Review: time-harmonic fields in source-free region
- The birth of waves?

- Wave equations
  - Homogeneous equations in time-domain

Time-domain Maxwell's equations

- Simple medium
  - Linear, homogeneous, isotropic: $\vec{D} = \varepsilon \vec{E}; \quad \vec{B} = \mu \vec{H}$
- Charge-free: $\rho = 0$
- Non-conducting: $\vec{J} = 0$

$$\begin{cases} \nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{D} = \rho \\ \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{B} = 0 \end{cases} \longrightarrow \begin{cases} \nabla \times \vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t} & (1) \\ \nabla \cdot \vec{E} = 0 & (2) \\ \nabla \times \vec{H} = \varepsilon \dfrac{\partial \vec{E}}{\partial t} & (3) \\ \nabla \cdot \vec{H} = 0 & (4) \end{cases}$$

Time-domain homogeneous wave equation-(1)

- Take curl of Eq. (1) $\nabla\times\vec{E}=-\mu\dfrac{\partial\vec{H}}{\partial t}$

charge free

$$\nabla\times\nabla\times\vec{E}=\nabla(\nabla\bullet\vec{E})-\nabla^2\vec{E}$$

$$\nabla\times\vec{H}=\varepsilon\frac{\partial\vec{E}}{\partial t}$$

$$=\nabla\times\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)=-\mu\frac{\partial}{\partial t}\left(\nabla\times\vec{H}\right)$$

M-homog.

$$=-\mu\frac{\partial}{\partial t}\left(\varepsilon\frac{\partial\vec{E}}{\partial t}\right)=-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\Rightarrow\nabla^2\vec{E}-\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}=0$$

Free-space: $\nabla^2\vec{E}-\dfrac{1}{c^2}\dfrac{\partial^2\vec{E}}{\partial t^2}=0$

Time-domain homogeneous wave equation-(2)

- Take curl of Eq. (3) $\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t}$

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\nabla \times \nabla \times \vec{H} = \nabla(\nabla \bullet \vec{H}) - \nabla^2 \vec{H}$$

$$= \nabla \times \left(\varepsilon \frac{\partial \vec{E}}{\partial t} \right) = \varepsilon \frac{\partial}{\partial t}\left(\nabla \times \vec{E} \right)$$

E-homog.

$$= \varepsilon \frac{\partial}{\partial t}\left(-\mu \frac{\partial \vec{H}}{\partial t} \right) = -\mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{H} - \mu\varepsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

So e-field and m-field have same velocity

Comments

- We assumed charge-free and current-free: $\rho = 0, \ \vec{J} = 0$
  - These equations only deals how the waves propagate.
  - They do not tell us how the waves are generated.

- We assume a simple medium. If the medium is complicated: (nonlinear, anisotropic, inhomogeneous), then the wave equation will be different.

- Time-harmonic fields (frequency-domain)
  - Wave equation with sinusoidal time functions
  - Helmholtz's equations

Why *time-harmonics*?

- Any periodic (aperiodic) function ➔ superposition of discrete (continuous) sinusoidal functions by Fourier series (integral).
- Maxwell's equations are linear.
  - Sinusoidal sources produce sinusoidal fields of the same frequency in steady state.
  - Total field can be derived by superposition of individual sinusoidal responses.
- Easy to operate if phasors are used: $\vec{A}(\vec{r},t)=\vec{A}(\vec{r})e^{j\omega t}$

$$\frac{\partial}{\partial t}\rightarrow j\omega,\quad \int dt\rightarrow\frac{1}{j\omega}$$

Scalar to vector phasor notation

- Scalar phasors of voltages & currents are sufficient to describe steady-state response of TX lines:

$$v(z,t) = \mathrm{Re}\left\{V(z)\cdot e^{j\omega t}\right\}$$

$$i(z,t) = \mathrm{Re}\left\{I(z)\cdot e^{j\omega t}\right\}$$

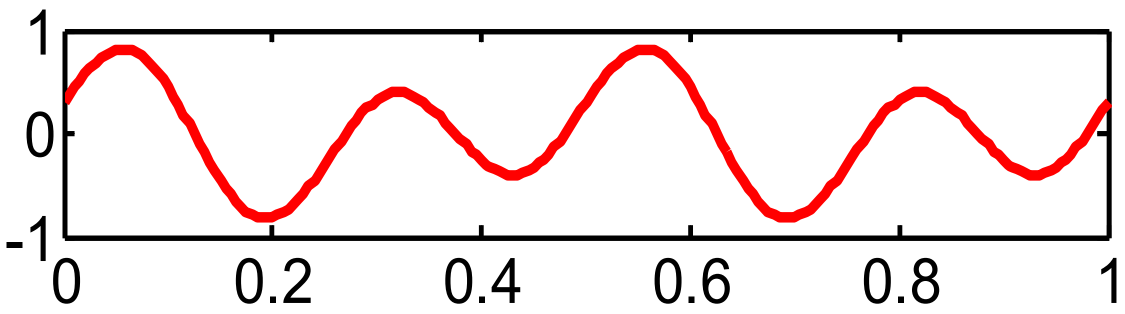
- Vector phasors of E-field and M-field are required to describe time-harmonic EM fields:

$$\vec{E}(x,y,z,t) = \mathrm{Re}\left\{\vec{E}(x,y,z)e^{j\omega t}\right\}$$

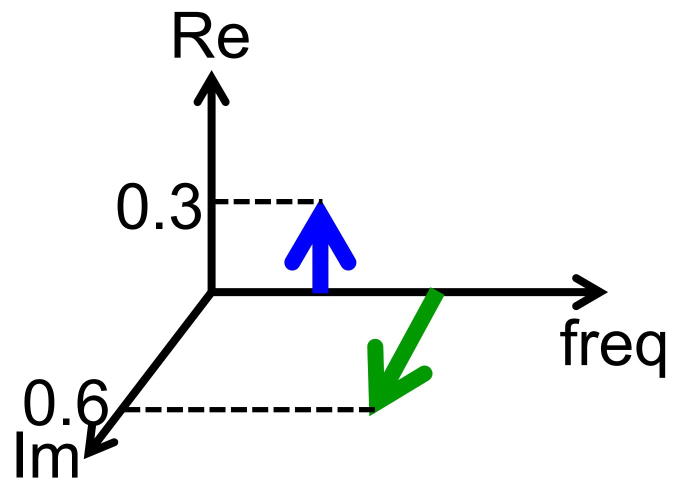
$$\vec{H}(x,y,z,t) = \mathrm{Re}\left\{\vec{H}(x,y,z)e^{j\omega t}\right\}$$

Phasor: time ➔ frequency

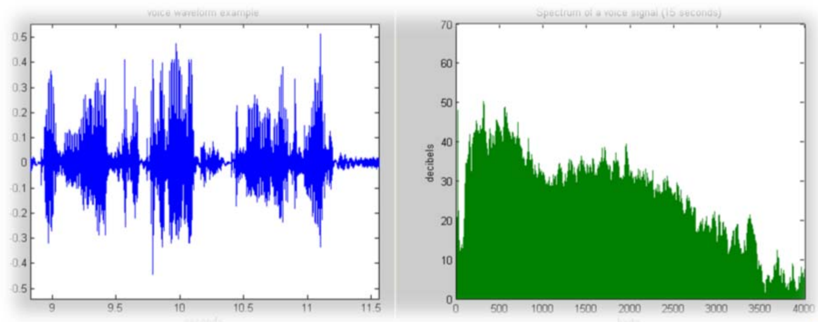
- Extract magnitude and phase frequency-by-frequency



$$e(t) = 0.3\cos(2\pi \times 2t) + 0.6\sin(2\pi \times 4t)$$



time



Frequency-domain Maxwell's equations

- For simple, source-free, current-free medium

$$\begin{cases} \nabla\times\vec{E} = -\mu\dfrac{\partial\vec{H}}{\partial t} \\ \nabla\cdot\vec{E} = 0 \\ \nabla\times\vec{H} = \varepsilon\dfrac{\partial\vec{E}}{\partial t} \\ \nabla\cdot\vec{H} = 0 \end{cases} \xrightarrow[\frac{\partial}{\partial t}\rightarrow j\omega]{\vec{E}(\vec{r},t)\rightarrow\vec{E}(\vec{r}),\ \vec{H}(\vec{r},t)\rightarrow\vec{H}(\vec{r})} \begin{cases} \nabla\times\vec{E} = -j\omega\mu\vec{H} & (1) \\ \nabla\cdot\vec{E} = 0 & (2) \\ \nabla\times\vec{H} = j\omega\varepsilon\vec{E} & (3) \\ \nabla\cdot\vec{H} = 0 & (4) \end{cases}$$

Frequency-domain wave equation-(1)

- Take curl of Eq. (1) $\nabla \times \vec{E} = -j\omega\mu\vec{H}$

$$\nabla \times \nabla \times \vec{E} = \nabla\left(\nabla \cdot \vec{E}\right) - \nabla^2 \vec{E} = -\nabla^2 \vec{E}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$$

$$= \nabla \times \left(-j\omega\mu\vec{H}\right) = -j\omega\mu\left(\nabla \times \vec{H}\right)$$

$$= -j\omega\mu\left(j\omega\varepsilon\vec{E}\right) = \omega^2 \mu\varepsilon\vec{E}$$

$$k \equiv \omega\sqrt{\mu\varepsilon} = \frac{\omega}{u_p} = \frac{2\pi}{\lambda}$$

$$\Rightarrow \nabla^2 \vec{E} + k^2 \vec{E} = 0$$

Wave vector, propagation constant

Frequency-domain wave equation-(2)

- Take curl of Eq. (3) $\nabla \times \vec{H} = j\omega\varepsilon\vec{E}$

$$\nabla \times \nabla \times \vec{H} = -\nabla^2 \vec{H}$$

$$\nabla \times \vec{E} = -j\omega\mu\vec{H}$$

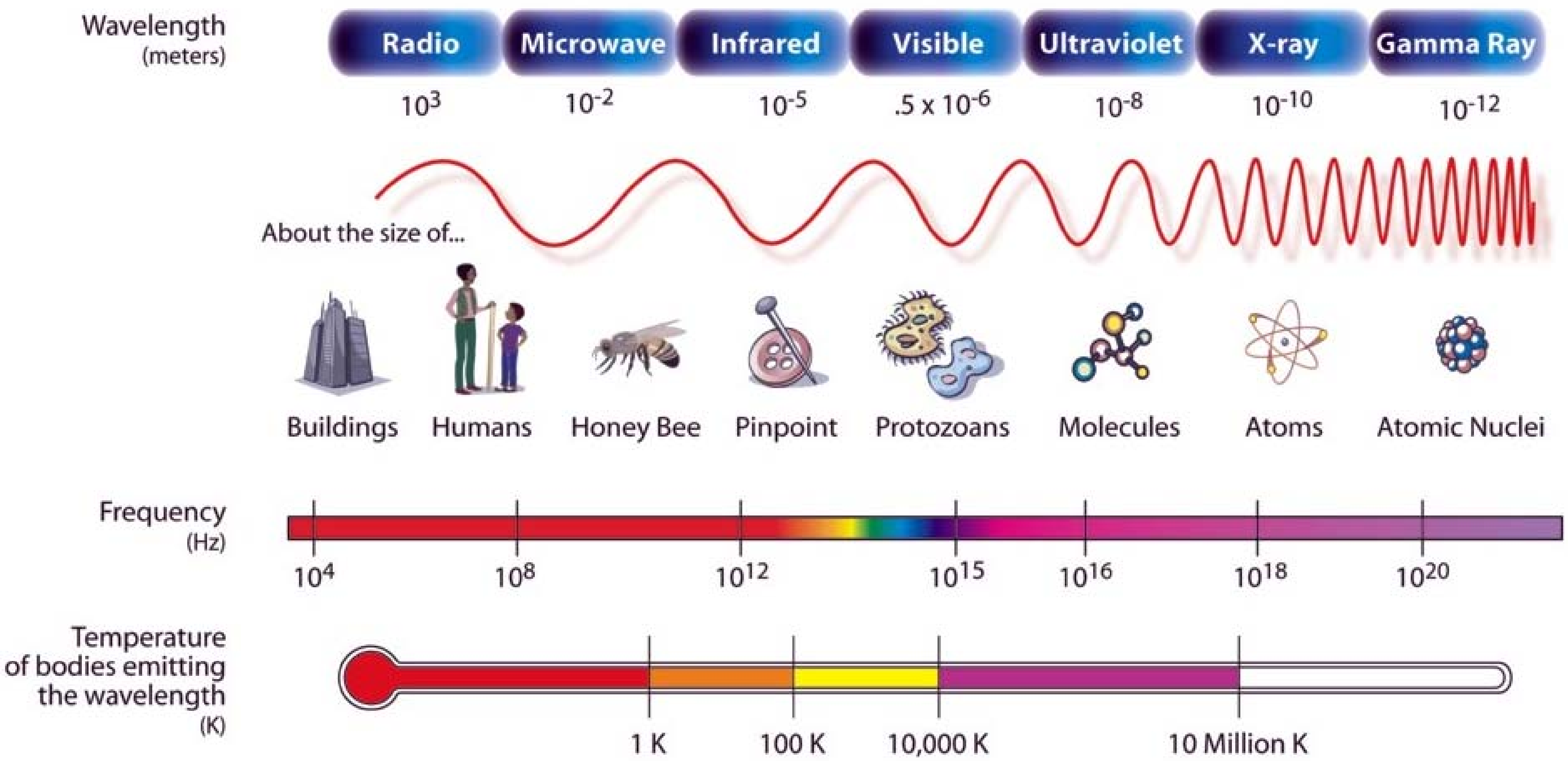
$$= \nabla \times \left(j\omega\varepsilon\vec{E} \right) = j\omega\varepsilon\left(\nabla \times \vec{E} \right)$$

$$= j\omega\varepsilon\left(-j\omega\mu\vec{H} \right) = \omega^2\mu\varepsilon\vec{H}$$

$$\Rightarrow \nabla^2 \vec{H} + k^2 \vec{H} = 0$$

The EM spectrum

- Can all be calculated by Maxwell's equations



- Radiation
  - Generation of waves
  - Brewster's angle revisited

Where does light come from?

The wave equation describes the propagation of light.

But where does light come from in the first place?

Some matter must emit the light.

It does so through the matter's polarization:

$$\vec{P}(t) = Nq\vec{x}_q(t)$$

Note that matter's polarization is analogous to the polarization of light.

N: the number density of charged particles

q: is the charge of each particle

$\vec{x}_q(t)$ is the position of the charge.

Assuming each charge is identical and has identical motion.

How to take medium into account?

- In vacuum

$$\vec{D} = \varepsilon_0 \vec{E}$$

- Now that we have polarization

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

- If polarization is taken as scalar

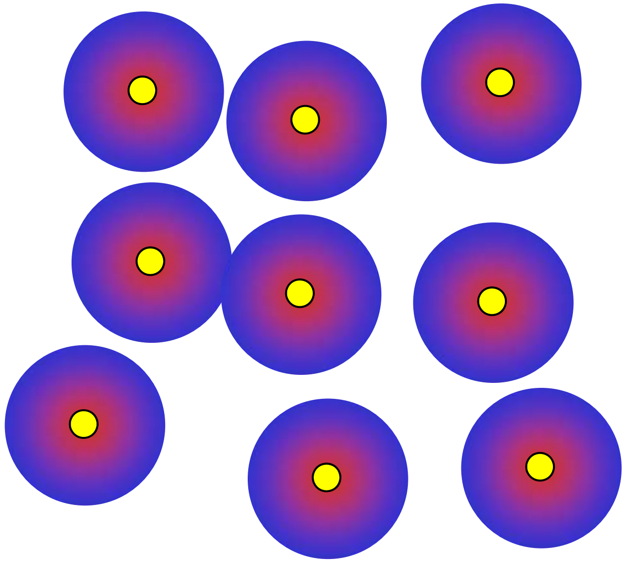
$$P = \varepsilon_0 (\chi^{(1)} E + h.o.t.)$$

$$D = \varepsilon_0 (1 + \chi^{(1)}) E$$

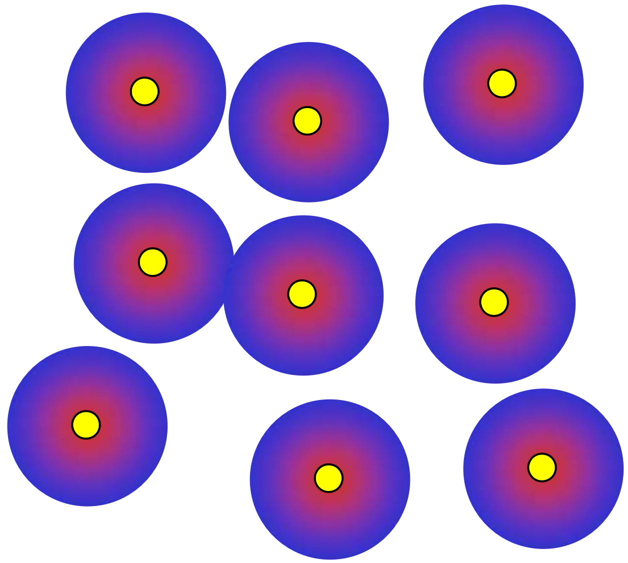
Polarized and unpolarized media

Unpolarized medium

(random phase)



Polarized medium



On the right, the displacements of the charges are correlated, so it is polarized at any given time.

Maxwell's equations for a medium

- The induced polarization, $\vec{P}$, contains the effect of the medium and is included in Maxwell's Equations:

$$\vec{\nabla} \cdot \vec{E} = 0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \frac{\partial \vec{P}}{\partial t}$$

This extra term turn it into the **Inhomogeneous Wave Equation**:

$$\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 N q \frac{\partial^2 \vec{x}_q}{\partial t^2}$$

The polarization is the **driving/source term** and tells us what light will be emitted.

But $\partial^2 \vec{x}_q / \partial t^2$ is just the charge acceleration!

So it's accelerating charges that emit light!

Sources of light

Accelerating charges emit light

- **Linearly accelerating charge**

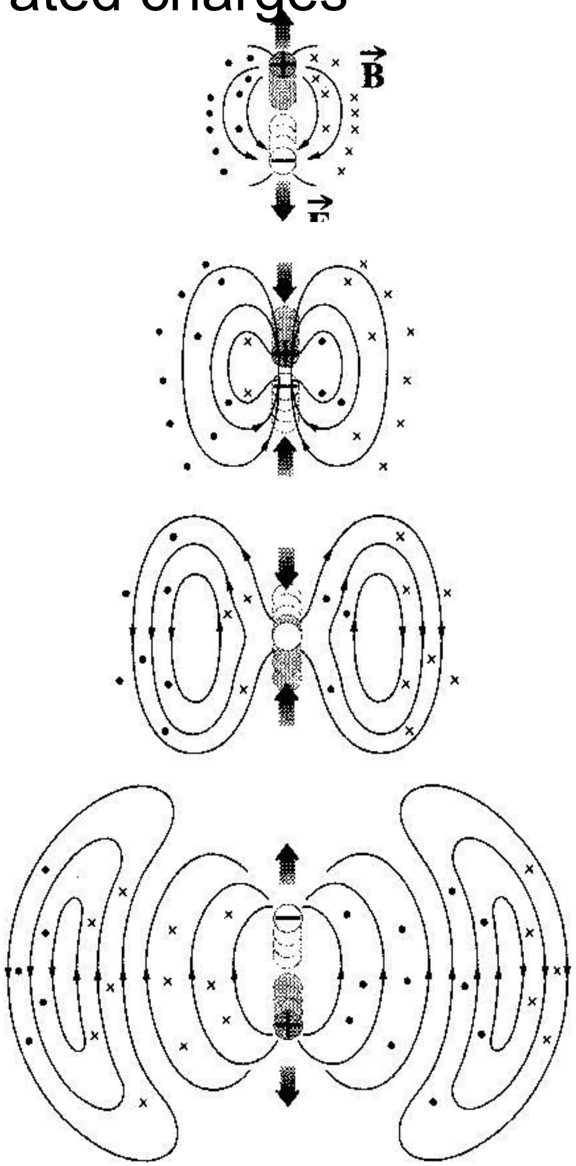
- Synchrotron radiation—light emitted by charged particles deflected by a magnetic field

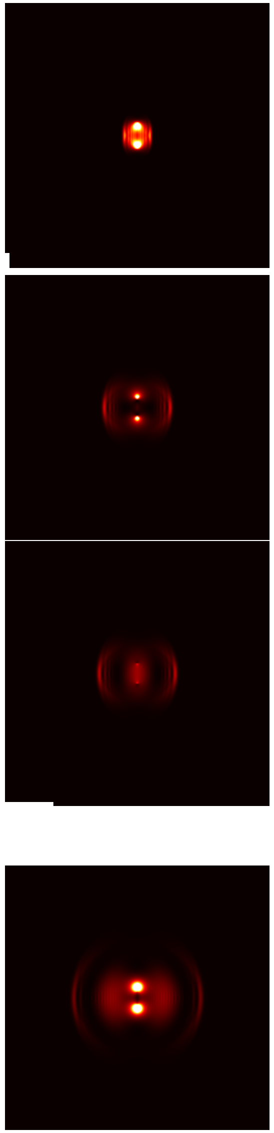
- Bremsstrahlung (Braking radiation)—light emitted when charged particles collide with other charged particles

Check: earthquake light!

Electric dipole radiation

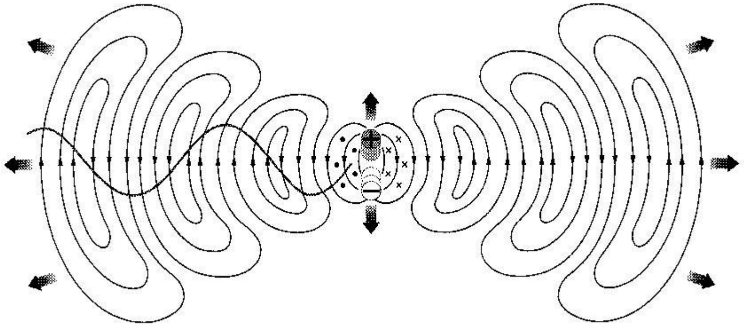
- Two spatially separated charges
  - One positive
  - One negative

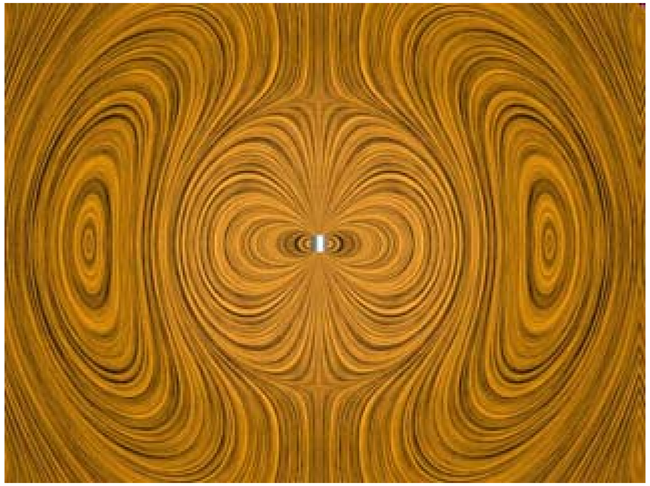


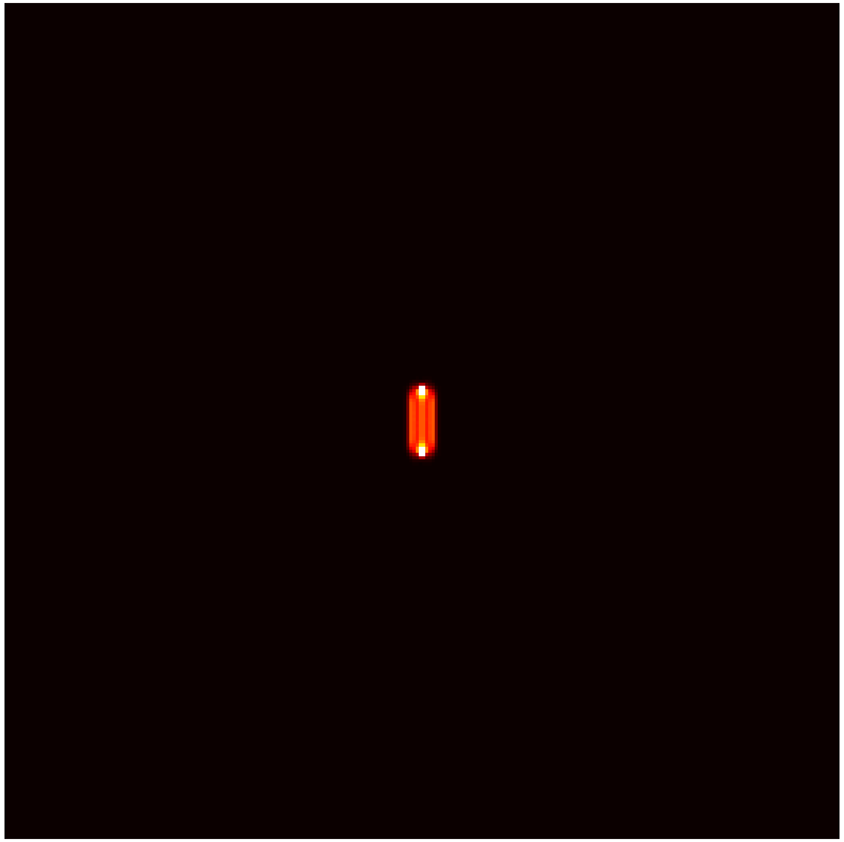


Electric dipole radiation-cont.

- Frequency, wavelength







Radiation pattern

- Toroidal pattern
- No field in the direction of motion!!
  - How to align the antennas of your wireless rout