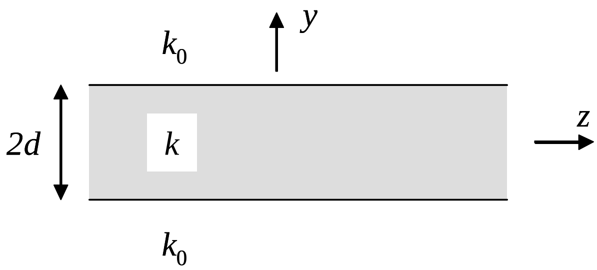
7.3 Dielectric Slab Waveguide

We will now examine the waveguide properties of a "slab" of dielectric material.



Two key concepts concerning dielectric waveguides deserve attention. The first is that, due to the symmetry of the geometry, the fields will either be symmetric or anti-symmetric about the x-z plane. The second is that in order for the field to be guided by the high-permittivity dielectric slab, the fields outside the slab must be evanescent, *i.e.* they decay in the y direction. What we will have inside the slab is a plane wave that bounces back and forth due to total internal reflection. We will use these observations in the formulations that follow.

7.3.1 TE Modes

The electric field for the TE modes must satisfy the homogeneous wave equation. Our experience with the parallel plate waveguide tells us what the solutions must be (before application of the boundary conditions). However, our argument about symmetry makes it so that within the slab, the variation will either be $\sin k_y y$ or $\cos k_y y$ (recall that in general, the field variation can be a combination of these two). Therefore, the electric field can be written as

$$\overline{E} = \hat{x}\begin{cases} E_1 e^{-\alpha y - jk_z z} & y \geq d \\ E_0 \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} e^{-jk_z z} & |y| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} E_1 e^{+\alpha y - jk_z z} & y \leq -d \end{cases} \tag{7.69}$$

where the top and bottom lines in the braces refer to the antisymmetric and symmetric modes, respectively. Using Faraday's law, we can now compute the magnetic fields

$$\overline{H} = \frac{-1}{j\omega\mu}\nabla\times\overline{E} = \begin{cases} \frac{E_1}{\omega\mu_0}\left(\hat{y}k_z + \hat{z}j\alpha\right) e^{-\alpha y - jk_z z} & y \geq d \\ \frac{E_0}{\omega\mu}\left(\hat{y}k_z \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} + \hat{z}jk_y \left\{ \begin{array}{r} -\cos k_y y \\ \sin k_y y \end{array} \right\}\right) e^{-jk_z z} & |y| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{E_1}{\omega\mu_0}\left(\hat{y}k_z - \hat{z}j\alpha\right) e^{+\alpha y - jk_z z} & y \leq -d \end{cases} \tag{7.70}$$

Note that in these field expressions, we have used that the z variation is $e^{-jk_z z}$ both inside and outside the slab. How do we know that this propagation constant is the same in both regions? Note also that we have four unknowns: E_1/E_0, k_y, α, and k_z.

Since these fields must obey the wave equation (with $\partial^2/\partial x^2 = 0$), we know that:

$$k_y^2 + k_z^2 = k_1^2 = \omega^2 \mu \epsilon_1 \tag{7.71}$$

$$-\alpha^2 + k_z^2 = k_2^2 = \omega^2 \mu_0 \epsilon_2 \tag{7.72}$$

which gives us two constraints for determining our unknowns. We need two additional constraints in order to find all four unknowns.

Let's start by enforcing continuity of tangential electric fields at the dielectric-air interface. We will first consider the symmetric modes. Therefore, at $y = d$

$$E_0 \cos(k_y d) e^{-jk_z z} = E_1 e^{-\alpha d} e^{-jk_z z} \qquad \rightarrow \qquad \cos(k_y d) E_0 = e^{-\alpha d} E_1 \tag{7.73}$$

Note that applying continuity at $y = -d$ results in an identical equation, so this does not help us. This stems from the symmetry of the problem, and in reality we have already used this symmetry to break the problem into symmetric and asymmetric modes.

Since we need one more equation, we will apply continuity of tangential (*i.e.* $\hat{z}$ component) magnetic field at the boundary. At $y = d$ we have

$$jk_y \frac{E_0}{\omega\mu} \sin(k_y d) e^{-jk_z z} = j\alpha \frac{E_1}{\omega\mu_0} e^{-\alpha d} e^{-jk_z z} \qquad \rightarrow \qquad \frac{k_y}{\mu} \sin(k_y d) E_0 = \frac{\alpha}{\mu_0} e^{-\alpha d} E_1 \tag{7.74}$$

and again, we get the exact same equation at $y = -d$. The easiest thing to do is to divide these two equations by each other. This yields

$$\frac{\frac{k_y}{\mu} \sin(k_y d) E_o}{\cos(k_y d) E_o} = \frac{\frac{\alpha}{\mu} e^{-\alpha d} E_1}{e^{-\alpha d} E_1} \tag{7.75}$$

$$\frac{\alpha}{\mu_0} = \frac{k_y}{\mu} \tan(k_y d). \tag{7.76}$$

which can be re-written as

$$\boxed{(\alpha d) = \frac{\mu_0}{\mu}(k_y d) \tan(k_y d) \qquad \text{symmetric TE modes}} \tag{7.77}$$

Combining (7.71) and (7.72) leads to

$$(k_y d)^2 + (\alpha d)^2 = \omega^2 \mu_0 \epsilon_0 (n_1^2 - n_2^2) d^2 \tag{7.78}$$

Finally, we can combine these two equations to the form

$$\tan(k_y d) = \sqrt{\frac{\omega^2 \mu_0 \epsilon_0 (n_1^2 - n_2^2) d^2}{(k_y d)^2} - 1} \tag{7.79}$$

1. Solutions in the range $(m-1)\pi/2 \leq k_y d \leq m\pi/2$ $m = 1, 3, 5, \ldots$ we will call TE$_m$ modes. These correspond to the symmetric TE modes.

2. Cutoff occurs when the mode is no longer guided, which occurs as soon as α becomes imaginary. So, we define cutoff as the frequency at which $\alpha = 0$. Using (7.77), this implies that $\tan(k_y d) = 0$ such that $k_y d = (m-1)\pi/2$, $m = 1, 3, 5, \ldots$. Using (7.78) with $\alpha = 0$ leads to
$$f_m = \frac{m-1}{4d} \frac{1}{\sqrt{\mu_0 \epsilon_0 (\mu_r \epsilon_r - 1)}} \tag{7.80}$$
Note that $f_1 = 0$, so the lowest order mode propagates at any frequency. Furthermore, since at cutoff $k_z = k_0$ and $k_z^2 + k_y^2 = k^2$, the angle of incidence of the wave on the dielectric boundary can be expressed as
$$\theta_i = \sin^{-1} \frac{k_z}{\sqrt{k_z^2 + k_y^2}} = \sin^{-1} \frac{k_0}{k} = \sin^{-1} \sqrt{\frac{\mu_0 \epsilon_0}{\mu \epsilon}} = \theta_c \tag{7.81}$$
which you may recognize at the critical angle. So, cutoff occurs when the angle of incidence on the boundary is smaller than the critical angle. Makes sense, doesn't it?

   Observe also that the cutoff condition of $k_z = k_0$ means that the propagation constant becomes that of the surrounding medium. We will revisit this below in optical fibers.

3. Note that k_y is frequency dependent, unlike in the parallel plate waveguide.

4. As the frequency gets larger, $\alpha \to \infty$ which means that the field decays very rapidly outside the dielectric. The behavior of the mode becomes like that of a parallel plate waveguide filled with a dielectric.

Note that we could repeat the entire procedure for the antisymmetric TE modes. The dispersion relation (7.78) remains the same. The guidance condition becomes

$$\boxed{(\alpha d) = -\frac{\mu_0}{\mu}(k_y d)\cot(k_y d) \qquad \text{antisymmetric TE modes}} \tag{7.82}$$

Again, cutoff occurs for $k_y d = (m-1)\pi/2$, $m = 2, 4, 6, \ldots$. These are therefore the even order TE modes.

We can solve these nonlinear transcendental equations using a nonlinear solver on a computer or calculator. We can also solve these equations graphically. We will plot each equation separately with plot axes of $k_y d$ and αd. Think of $k_y d \equiv x$ and $\alpha d \equiv y$. The various equations then become

$$(\alpha d) = \frac{\mu_0}{\mu}(k_y d)\tan(k_y d) \qquad \text{symmetric TE modes} \tag{7.83}$$
$$y = x\tan(x), \tag{7.84}$$

$$((\alpha d) = -\frac{\mu_0}{\mu}(k_y d)\cot(k_y d) \qquad \text{antisymmetric TE modes} \tag{7.85}$$
$$y = -x\cot(x) \tag{7.86}$$

and

$$(k_y d)^2 + (\alpha d)^2 = \omega^2 \mu_o \epsilon_o \left(n_1^2 - n_2^2\right) d^2 \tag{7.87}$$
$$x^2 + y^2 = (k_o d)^2 \left(n_1^2 - n_2^2\right) \tag{7.88}$$

7.3.2 TM Modes

We can repeat the whole process for TM modes. In this case, we have

$$\overline{H} = \hat{x} \begin{cases} H_1 e^{-\alpha y - jk_z z} & y \geq d \\ H_0 \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} e^{-jk_z z} & |y| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} H_1 e^{+\alpha y - jk_z z} & y \leq -d \end{cases} \tag{7.89}$$

where the top and bottom lines in the braces refer to the antisymmetric and symmetric modes, respectively. Using Ampere's law, we can now compute the electric fields

$$\overline{E} = \frac{1}{j\omega\epsilon} \nabla \times \overline{H} = \begin{cases} \frac{H_1}{\omega\epsilon_0} \left(-\hat{y}k_z - \hat{z}j\alpha\right) e^{-\alpha y - jk_z z} & y \geq d \\ \frac{H_0}{\omega\epsilon} \left(-\hat{y}k_z \left\{ \begin{array}{c} \sin k_y y \\ \cos k_y y \end{array} \right\} - \hat{z}jk_y \left\{ \begin{array}{r} -\cos k_y y \\ \sin k_y y \end{array} \right\} \right) e^{-jk_z z} & |y| \leq d \\ \left\{ \begin{array}{c} - \\ + \end{array} \right\} \frac{H_1}{\omega\epsilon_0} \left(-\hat{y}k_z + \hat{z}j\alpha\right) e^{+\alpha y - jk_z z} & y \leq -d \end{cases} \tag{7.90}$$

We go through the exact same sequence of steps for this case. The dispersion relations remain the same. The guidance conditions become

$$\boxed{\begin{aligned} (\alpha d) &= \frac{\epsilon_0}{\epsilon}(k_y d)\tan(k_y d) \qquad \text{symmetric TM modes} && (7.91) \\ (\alpha d) &= -\frac{\epsilon_0}{\epsilon}(k_y d)\cot(k_y d) \qquad \text{antisymmetric TM modes} && (7.92) \end{aligned}}$$