The solutions

- The solutions to the wave equation for the *three* regions have the form (***assuming TE polarization***)

$$E = \hat{y}E_y(x)\exp i(\beta z - \omega t)$$

- Substituting into the wave equation (*assuming no y dependence*)

$$\frac{\partial^2 E_y}{\partial x^2} + (k_i^2 - \beta^2)E_y = 0$$

- We can solve the second-order differential equation (*with constant coefficients*) for E_y

$$E_y \propto \exp\left(\pm ix\sqrt{k_i^2 - \beta^2}\right)$$

Symmetric slab waveguide solutions

- The solutions are *sinusoidal* or *exponential* according to

Sinusoidal	$k_i^2 > \beta^2$
Exponential	$k_i^2 < \beta^2$

- **Guided modes**: cladding region has *exponential* solutions $\beta > k_2 > k_3$ while core region has *sinusoidal* solutions $\beta < k_1$
- For $x \rightarrow \pm\infty$, we require the solutions to remain finite.
- The solutions have the general form (*assuming* $n_2 = n_3$):

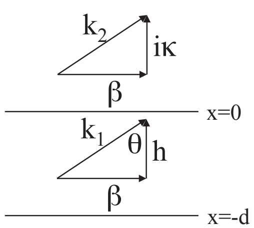
Cladding ($x \geq 0$)	$E_y = A\exp(-\kappa x)$
Core ($-d \leq x \leq 0$)	$E_y = B\cos(hx) + C\sin(hx)$
Cladding ($x \leq -d$)	$E_y = D\exp(\kappa(x+d))$

Symmetric slab waveguide solutions

- The *transverse* propagation constants

$$\kappa = (\beta^2 - k_2^2)^{1/2}$$

$$h = (k_1^2 - \beta^2)^{1/2} = k_1\cos\theta$$



- In order to determine the *allowed* β and the unspecified constants A, B, C and D, we need to match the solution in cladding with the solution in core.
- Therefore, ***boundary conditions*** must be specified at the core-cladding interfaces.
- We expect at least *one* arbitrary constant in the final solution given by the overall field strength.

Boundary conditions

- E_y *continuous* at $x = 0 \rightarrow A = B$
- $\partial E_y/\partial x$ (H_z) *continuous* at $x = 0 \rightarrow C = (-\kappa/h)A$
- E_y *continuous* at $x=-d \rightarrow D = A\cos(hd) + (\kappa A/h)\sin(hd)$
- $\partial E_y/\partial x$ (H_z) *continuous* at $x=-d \rightarrow \tan(hd) = 2\kappa h / (h^2 - \kappa^2)$

The first *three* results can be used to solve the ***electric field distributions*** within the waveguide core and cladding.

The *fourth* result gives a *transcendental* equation that allows us to solve for the ***allowed β*** *graphically*.

TE mode field distributions in a symmetric slab waveguide

$x \geq 0 \qquad E = \hat{y}A\exp(-\kappa x)\exp i(\beta z - \omega t)$

$-d \leq x \leq 0 \qquad E = \hat{y}A[\cos(hx) - \frac{\kappa}{h}\sin(hx)]\exp i(\beta z - \omega t)$

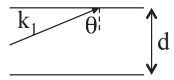
$x \leq -d \qquad E = \hat{y}A\exp(\kappa(x+d))[\cos(hd) + \frac{\kappa}{h}\sin(hd)]\exp i(\beta z - \omega t)$

- Note that *all* the electric field distributions propagate along the z-direction because of the factor exp i($\beta z - \omega t$) even though the cladding field decays exponentially in the transverse direction.
- Next we solve for the allowed β, κ and h.

Eigenvalue equations for symmetric slab waveguides

- The ***transverse resonance condition*** for a *symmetric* waveguide

$$2k_1 d \cos\theta + 2\varphi(\theta) = 2m\pi$$



- The *reflection phase angle* for the TE polarization (s wave) is given as (see lecture 1)

$$\tan(\varphi_{TE}/2) = (n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

- The *reflection phase angle* for the TM polarization (p wave) is given as

$$\tan(\varphi_{TM}/2) = (n_1^2/n_2^2)(n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

Eigenvalue equations for symmetric slab waveguides

Recall the reflection coefficient $r_{TE} = \exp -i\varphi_{TE}$ => $\varphi(\theta) = -\varphi_{TE}(\theta)$

$r_{TM} = -\exp -i\varphi_{TM}$ => $\varphi(\theta) = -\varphi_{TM}(\theta)$

=> $\quad k_1 d \cos\theta - m\pi = -\varphi(\theta) = \varphi_{TE,TM}(\theta)$

=> $\quad (k_1 d \cos\theta)/2 - m\pi/2 = \varphi_{TE,TM}/2$

=> $\quad \tan[(k_1 d \cos\theta)/2 - m\pi/2] = \tan(\varphi_{TE,TM}/2)$

TE:

$$\tan[(k_1 d \cos\theta)/2 - m\pi/2] = (n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

TM:

$$\tan[(k_1 d \cos\theta)/2 - m\pi/2] = (n_1^2/n_2^2)(n_1^2 \sin^2\theta - n_2^2)^{1/2} / (n_1 \cos\theta)$$

$m = 0, 1, 2, \ldots,$

Normalized waveguide parameters

- The mode properties of a waveguide are commonly characterized in terms of *dimensionless* normalized waveguide parameters.
- The ***normalized frequency***, also known as the ***V number***, of a step-index planar waveguide is defined as

$$V = (2\pi/\lambda)\, d\, (n_1^2 - n_2^2)^{1/2} = (\omega/c)\, d\, (n_1^2 - n_2^2)^{1/2}$$

where d is the core thickness (*or diameter in the case of optical fibers*).

- The propagation constant β can be represented by the ***normalized guide index***:

$$b = (\beta^2 - k_2^2)/(k_1^2 - k_2^2) = (n_{eff}^2 - n_2^2)/(n_1^2 - n_2^2)$$

recall $n_{eff} = c\beta/\omega = \beta\lambda/2\pi$ is the *effective refractive index* of the waveguide mode that has a propagation constant β.

Eigenvalue equations in terms of normalized frequency

TE: $\tan(hd/2 - m\pi/2) = (V^2 - h^2d^2)^{1/2}/hd$

TM: $\tan(hd/2 - m\pi/2) = (n_1^2/n_2^2)(V^2 - h^2d^2)^{1/2}/hd$

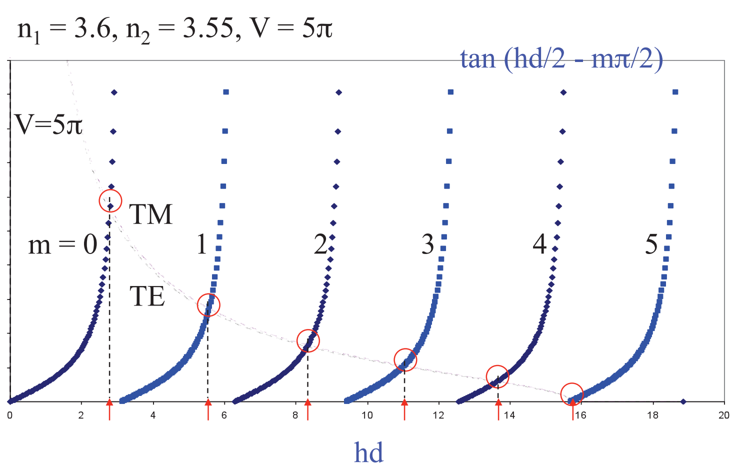
$m = 0, 1, 2, \ldots,$

- The eigenvalue equations are in the form of *transcendental* equations, which are usually solved graphically by plotting their left- and right-hand sides as a function of hd.
- The solutions yield the *allowed values of hd* for a given value of the waveguide parameter V for TE/TM modes.

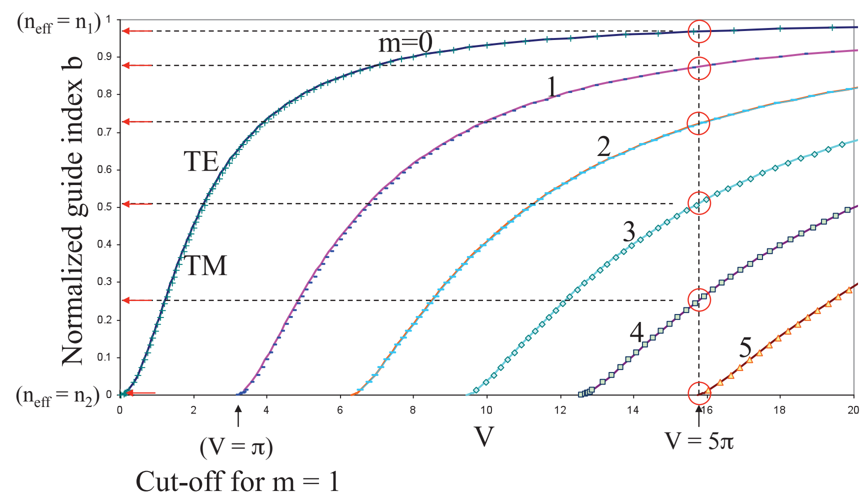
Example #1: Symmetric weakly guiding slab waveguides

- Consider a ***weakly guiding*** waveguide $n_1 - n_2 << n_1$
- Here we choose $n_1 = 3.6$ and $n_2 = 3.55$. These values are characteristic of an AlGaAs double heterojunction light-emitting diode or laser diode.
- The critical angle for this structure is $\theta_c = \sin^{-1}(n_2/n_1) \sim 80^o$
- The range of angles for trapped rays is then $80^o \leq \theta \leq 90^o$.
- The range of *waveguide effective refractive index* is $3.55 \leq n_{eff} \leq 3.6$

Graphic solutions for the eigenvalues of guided TE and TM modes of a weakly guiding symmetric slab waveguide



Mode chart for the first six TE and TM modes ($m = 0 - 5$) of symmetric slab waveguides in AlGaAs ($n_1 = 3.6$, $n_2 = 3.55$)

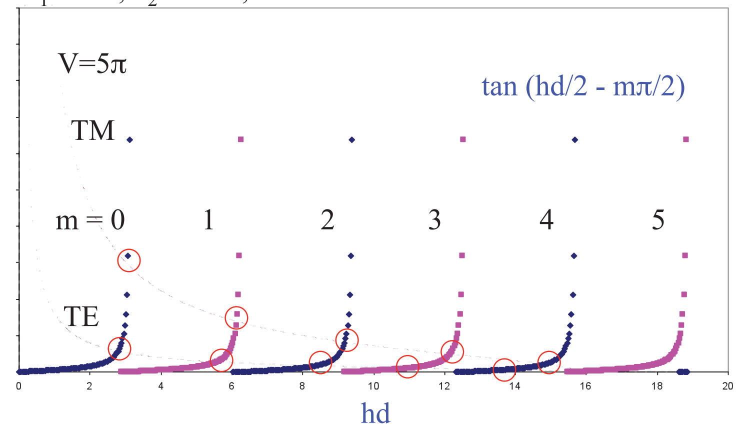


Example #2: Symmetric strongly guiding slab waveguides

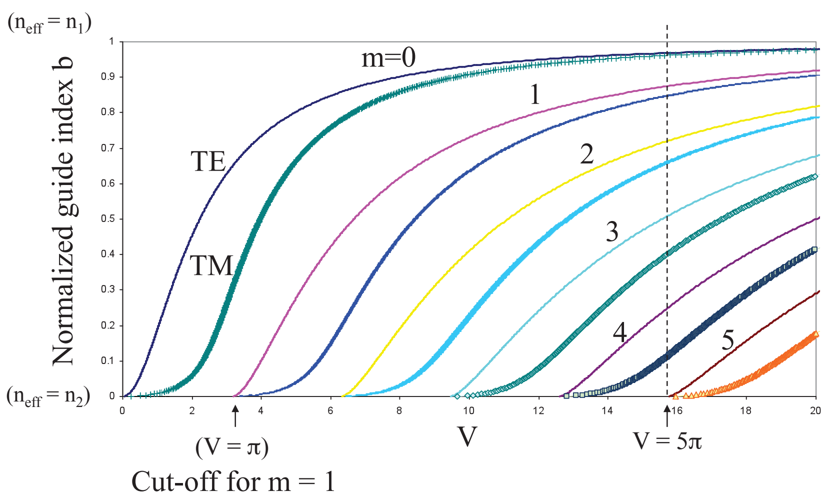
- Consider a ***strongly guiding*** waveguide $n_1 - n_2 >> 0$
- Here we choose $n_1 = 3.5$ and $n_2 = 1.45$. These values are characteristic of an silicon-on-insulator (SOI) waveguide.
- The critical angle for this structure is $\theta_c = \sin^{-1}(n_2/n_1) \sim 24.5^o$
- The range of angles for trapped rays is then $24.5^o \leq \theta \leq 90^o$.
- The range of *waveguide effective refractive index* is $1.45 \leq n_{eff} \leq 3.5$

Graphic solutions for the eigenvalues of guided TE and TM modes of a strongly guiding symmetric slab waveguide

$n_1 = 3.5$, $n_2 = 1.45$, $V = 5\pi$



Mode chart for the first six TE and TM modes (m = 0 – 5) of symmetric slab waveguides in SOI ($n_1 = 3.5$, $n_2 = 1.45$)



Normalized guide index vs. V number

- When the V number is *very small* (e.g. $d/\lambda << 1$), the guided ray travels *close to the critical angle* ($b << 1$). The effective index is close to that of the cladding layer n_2.

=>The wave penetrates deeply into the cladding layers, because the rays are near the critical angle. The evanescent decay is slow.

- As the V number increases, the ray travels more nearly parallel to the waveguide axis, and the effective refractive index lies between n_1 and n_2.
- For a *very large* V number (e.g. $d/\lambda >> 1$) the effective index is near that of the core index n_1. The wave in the cladding layer decays very rapidly for evanescent waves traveling at angles far above the critical angle.

Cutoff conditions

- For example, consider $V = 15$ on the mode chart, the TE_5/TM_5 modes could not propagate because V was not large enough to intersect with the b vs. V curves.

=> The TE_5/TM_5 modes, and *all higher-ordered modes*, are *cut off*.

- *Cutoff* occurs when the propagation angle for a given mode just equals the *critical angle* θ_c --- a guided mode transits to an *unguided* radiation mode.

- This corresponds to the condition that $\beta = k_2$ ($b = 0$) and $\kappa = 0$.

- The fields *extend to infinity* for $\kappa = 0$ (i.e. the fields become unguided!). This defines the *cutoff condition* for guided modes.

Cutoff conditions

- The cutoff value $V = V_c$ for a particular guided mode is the value of V at the point where $b = 0$ (i.e. the b vs. V relation intersects with the axis $b = 0$).
- For $\kappa = 0$,

$$V^2 \equiv k_1^2 d^2 - k_2^2 d^2 = k_1^2 d^2 - \beta^2 d^2 = h^2 d^2$$

$$\Rightarrow \quad V_c = hd$$

Substitute this relation to the TE/TM modes eigenvalue equations:

$$\tan(hd/2 - m\pi/2) = 0$$

$$\Rightarrow V_c = m\pi, \; m = 0, 1, 2, \ldots$$

Number of modes

- TE and TM modes of a *symmetric* waveguide have the *same* cutoff condition:

$$V_c = m\pi \qquad \text{for the } m_{th} \text{ TE and TM modes}$$

- Because cutoff of the fundamental mode (TE_0/TM_0) occurs at *zero* thickness ($V = 0$), *neither fundamental TE nor fundamental TM mode in a symmetric waveguide has cutoff*.

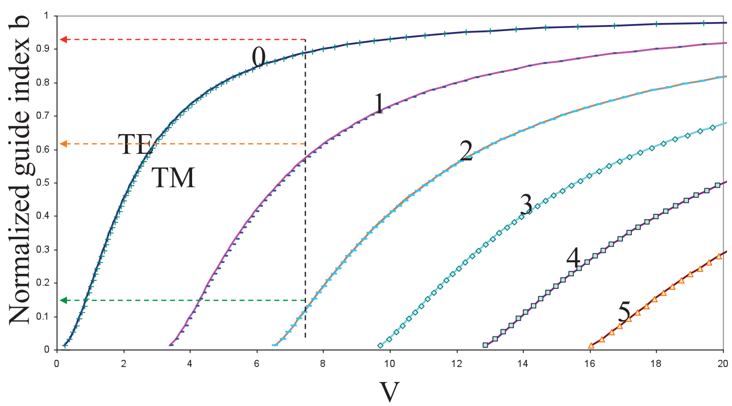
  => Any symmetric dielectric waveguide supports at least one TE and one TM mode.
- The number of TE modes supported by a given symmetric waveguide is the same as that of the TM modes and is

$$M_{TE} = M_{TM} = [V/\pi]_{integer}$$

— Nearest integer larger than the bracket value

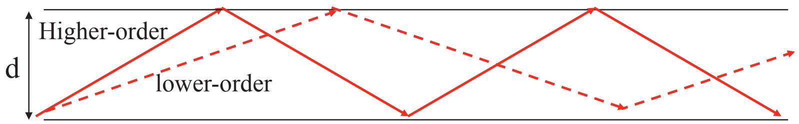
Number of modes

e.g. Find the number of TE/TM modes in an AlGaAs waveguide if $d = 1.64$ µm. The free-space wavelength is $\lambda = 0.82$ µm. $V = (2\pi d/\lambda)(n_1^2 - n_2^2)^{1/2} \approx 7.5$. $M_{TE} = M_{TM} = [7.5/\pi]_{int} = 3$. For this value, the mode chart yields three solutions – TE_0/TM_0, TE_1/TM_1, and TE_2/TM_2.



Singlemode vs. multimode waveguides

- A multimode waveguide is one that supports more than one propagating mode.
- For a multimode waveguide, at a fixed thickness the *higher*-ordered modes propagate with *smaller* β values than the lower-ordered modes.



*If we wish to propagate only the TE_0/TM_0 mode, then we must have

$$V < \pi \qquad \text{single-mode condition}$$

=> This *cuts off* the m = 1 mode and *all* higher-order modes.

Further remarks on TE/TM mode charts

- For the *weakly guiding* AlGaAs waveguide, TE and TM modes have about the same *effective index* and *propagation angle*, but their electric-field vectors point in orthogonal directions.

- Two *modes having the same propagation constant* β are said to be *degenerate*. In the example of AlGaAs waveguide, TE and TM modes of the same order are *nearly* degenerate.

- Even when n_1 is *not* close to n_2 (i.e. high-index contrast), the cutoff values for the TE_m and TM_m mode are still the same ($V_c = m\pi$ *still* applies for the TE/TM modes in high-index contrast waveguides.)

$\Rightarrow$The number of propagating TM modes equals that of the TE modes. (i.e. the *total* number of allowed modes is twice the number of modes found from the equation $[V/\pi]_{int}$)

Remarks on modes in asymmetric slab waveguides

- Cutoff of the fundamental mode (TE_0/TM_0) does *not* occur at zero thickness (V = 0), as it does for the symmetric case.
- Because the core n_1 and the cladding n_2 and n_3 are all different, the TE and TM modes are *not* degenerate. A *truly singlemode waveguide exists* if the TM_0 mode (but not the TE_0 mode) is cut off.

- ***Integrated optic circuits*** often adopt *asymmetric waveguide structures* and normally *singlemode waveguiding is desirable* (the *tradeoff* is that the waveguides become polarization dependent and only one polarization mode propagates!)

- Mode patterns for the asymmetric waveguide are similar to those of the symmetric waveguide, except that the asymmetry causes the fields to have *unequal* amplitudes at the two boundaries and to decay at different rates in the two cladding layers.

General formalisms for step-index planar waveguides

for asymmetric and symmetric slab waveguides